

# Emergent gravity in two dimensions

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We explore models with emergent gravity and metric by means of numerical simulations. A particular type of two-dimensional non-linear sigma-model is regularized and discretized on a quadratic lattice. It is characterized by lattice diffeomorphism invariance which ensures in the continuum limit the symmetry of general coordinate transformations. We observe a collective order parameter with properties of a metric, showing Minkowski or euclidean signature. The correlation functions of the metric reveal an interesting long-distance behavior with power-like decay. This universal critical behavior occurs without tuning of parameters and thus constitutes an example of “self-tuned criticality” for this type of sigma-models. We also find a non-vanishing expectation value of a “zweibein” related to the “internal” degrees of freedom of the scalar field, again with long-range correlations. The metric is well described as a composite of the zweibein. A scalar condensate breaks euclidean rotation symmetry.

## I. INTRODUCTION

There are many attempts to formulate a quantum field theory for gravity by use of a lattice regularization. Regge-Wheeler lattice gravity[1] employs the lengths of edges of simplices as basic degrees of freedom and therefore uses directly elements of (discrete) geometry. Different geometrical objects are used in other formulations of lattice gravity[2, 3]. Lattice spinor gravity[4] is an example where the basic degrees of freedom are fermions, while no basic geometrical objects are introduced. An extension of lattice spinor gravity uses in addition to the spinors a geometrical field, namely a connection[5].

We follow here the approach that the metric is obtained as the expectation value of a suitable collective field, while geometrical quantities are not used as fundamental degrees of freedom[6, 7]. In this sense gravity and geometry emerge from a model of other, non-geometrical fields. In our approach, the decisive ingredient is diffeomorphism symmetry in the continuum limit. In four dimensions, diffeomorphism symmetry entails under rather general circumstances the presence of a massless spin-two particle, and therefore of a metric and the corresponding geometry. We employ a lattice formulation with the property of lattice diffeomorphism invariance [8] of the action and functional measure. This induces diffeomorphism symmetry whenever the model exhibits non-trivial long range physics which allows to formulate a continuum limit.

Lattice diffeomorphism invariant models have been proposed using fermions as fundamental degrees of freedom - namely lattice spinor gravity [4]. A different approach is formulated as a non-linear  $\sigma$ -model [8]. This latter approach offers the important advantage that relatively cheap numerical simulations can be used in order to compute expectation values and correlation functions of the collective metric field. The purpose of the present paper is a first numerical study of such lattice diffeomorphism invariant non-linear  $\sigma$ -models.

Our first approach concentrates on two dimensions. Sometimes it is said that two-dimensional gravity is triv-

ial since it does not exhibit a propagating degree of freedom. This statement holds, however, only if the quantum effective action is given by an Einstein-Hilbert term proportional to the curvature scalar  $R$ , plus a cosmological constant. Indeed, in two dimensions  $R$  is a topological invariant which cannot provide a kinetic term for the metric. We emphasize that there is no reason to believe that the quantum effective action for a composite metric should be purely of the Einstein-Hilbert type. Propagating metric degrees of freedom become possible for a different form of the quantum effective action. We demonstrate this in appendix A with a rather simple possible form of a diffeomorphism symmetric effective action. Similarly, we give in appendix B an example for a diffeomorphism symmetric effective action for a propagating zweibein. The effective action for the model investigated in the present paper turns out to be more complicated than these simple examples.

In this paper we investigate a non-linear  $\sigma$ -model with two complex scalar fields  $\varphi_i$ , with “flavor” index  $i = 1, 2$ . The constraint

$$\sum_i \varphi_i^* \varphi_i = 1 \quad (1)$$

is compatible with an  $SO(4)$ -flavor symmetry acting on the four real fields  $\varphi_{i,R}, \varphi_{i,I}, \varphi_i = \varphi_{i,R} + i\varphi_{i,I}$ . We consider the real action ( $x^\mu = (x^0, x^1)$ )

$$S = \beta \int d^2 x \epsilon^{\mu\nu} (\varphi_1^* \partial_\mu \varphi_1 - \varphi_1 \partial_\mu \varphi_1^*) (\varphi_2^* \partial_\nu \varphi_2 - \varphi_2 \partial_\nu \varphi_2^*), \quad (2)$$

with  $\epsilon^{01} = -\epsilon^{10} = 1$ ,  $\epsilon^{00} = \epsilon^{11} = 0$ . For continuous space this action is invariant under general coordinate transformations. No metric is introduced a priori - diffeomorphism symmetry is realized by the particular contraction of two derivatives with the  $\epsilon$ -tensor.

We discretise the action (2) on a two dimensional quadratic lattice by using lattice derivatives and cell averages as explained in detail in section II. We perform a numerical study of this model by Monte Carlo technics. The continuum diffeomorphism symmetry of the action implies diffeomorphism symmetry of the discretised lattice action.

However, we stress that the effective action describing the system in the continuum limit is not of the simple form (2).

Our main findings are the following:

- (i) We identify collective fields that have the transformation properties of a metric. They acquire indeed non-vanishing expectation values. There are several candidates for collective metric fields. For the vacuum we find non-vanishing flat metrics both with Minkowski and euclidean signature. The metric turns out to describe flat space independently of the parameters of the model, which raises interesting questions concerning a self-adjustment of the effective two-dimensional cosmological constant to zero.
- (ii) The correlations of the metric fluctuations are long range. They typically show a powerlike decay  $\sim r^{-\alpha}$ , with  $\alpha$  close to two.
- (iii) The geometry differs from flat space if sources corresponding to an energy momentum tensor are introduced. In the linear approximation the perturbation of the metric in response to a source is determined by the correlation function. A static point source, which mimicks a static massive object, leads to a “Newtonian” potential that decays with the inverse of the spatial distance. This is similar to four dimensional gravity, but quite different from naive dimension estimates.
- (iv) The critical behavior associated to the powerlaw for the correlation functions occurs independently of the detailed values of the parameter  $\beta$  characterizing the model. The non-linear  $\sigma$ -model (2) is an example for self-tuned criticality.
- (v) We also identify a collective zweibein  $e_\mu^m$ , where the “Lorentz index”  $m$  is associated to the flavor degrees of the scalar field. To a good approximation, some of the metric candidates can be described as the usual bilinear of the zweibein.
- (vi) The correlation functions of the zweibein decay with a power of the distance, similar to the metric.
- (vii) We identify the exact ground state of the model for  $\beta \rightarrow \infty$ . It consists of “stripe-configurations” for the scalar field.
- (viii) The stripes persist for  $\beta > \beta_c$ ,  $\beta_c \approx 4.6$ . They also characterize the vacuum state of a corresponding continuum theory. The scalar order parameters describing the stripes are responsible for a spontaneous breaking of euclidean rotation symmetry. For this reason we encounter, in case of euclidean signature, an unusual version of  $d = 2$  gravity. While diffeomorphism symmetry is expected to be realized in the continuum limit this does not hold for rotation symmetry, due to the presence of preferred axes.
- (ix) At  $\beta_c$  we find a first order transition. The disordered phase for  $\beta < \beta_c$  shows no stripes and no expectation

value for the zweibein. The correlations are short range in this phase.

Our paper is organized as follows. In section II we describe the discretisation of our model on a lattice. We define scalar order parameters and collective metric fields in sect. III. In section IV we show that the metric correlation functions have a long range, power law behaviour. The stripe configurations that dominate the system for large values of  $\beta$  are discussed in sect. V. We find the exact “ground state” of the system for  $\beta \rightarrow \infty$ . It is a stripe configuration for the scalar fields. In section VI we define and describe the zweibein as a bilinear in the scalar fields. The “internal index” or “Lorentz-index” of the zweibein is related to the flavor structure for the scalars. In section VII we discuss the presence of an approximate Lorentz symmetry of our non-linear  $\sigma$ -model. While the action (2) is Lorentz symmetric, the constraint (1) violates Lorentz symmetry. We describe the metric and zweibein correlation functions in sections VIII and IX, respectively. Finally, we conclude in section X.

The results presented in this paper are all obtained by numerical simulations. In parallel, we present in a series of appendices our analytical investigations of the possible form of effective action for two-dimensional gravity. These analytical considerations all apply to the continuum limit and exploit the symmetries of our model. We proceed on various levels: App. A discusses the metric as a unique degree of freedom, while App. B uses the zweibein. In App. D we investigate the effective action for the scalar fields, and App. E combines scalars and the zweibein. Only the effective actions in App. D and E describe parts of our numerical findings in a realistic way. Finally, App. C discusses a possible connection between the correlation functions for the zweibein and the one for the scalars.

## II. LATTICE ACTION

We regularize our model on a lattice,  $x^0 = \tilde{z}^0 \Delta$ ,  $x^1 = \tilde{z}^1 \Delta$ , with  $\tilde{z}^\mu$  integers such that the sum  $\tilde{z}^0 + \tilde{z}^1$  is odd. This corresponds to a square lattice with lattice distance  $\sqrt{2}\Delta$  and nearest neighbors in the diagonal directions  $\sim x^0 \pm x^1$ . For the corresponding “diagonal lattice vectors” the  $\tilde{z}$ -coordinates are

$$E_0 = (1, 1), \quad E_1 = (-1, 1). \quad (3)$$

We define cells located on the sites of the dual lattice at  $x^\mu = \tilde{y}^\mu \Delta$ , with  $\tilde{y}^\mu$  integer and  $\tilde{y}^0 + \tilde{y}^1$  even. Each cell consists of four lattice points with lattice coordinates  $\tilde{z}^\mu = \tilde{y}^\mu \pm (\tilde{v}_\nu)^\mu$ , with two unit vectors  $\tilde{v}_\nu$  obeying  $(\tilde{v}_\nu)^\mu = \delta_\nu^\mu$ . The lattice derivative at  $\tilde{y}$  is given by

$$\partial_\mu \varphi(\tilde{y}) = \frac{1}{2\Delta} [\varphi(\tilde{y} + v_\mu) - \varphi(\tilde{y} - v_\mu)], \quad (4)$$

and cell averages obey

$$\bar{\varphi}(\tilde{y}) = \frac{1}{4} [\varphi(\tilde{y} + v_0) + \varphi(\tilde{y} - v_0) + \varphi(\tilde{y} + v_1) + \varphi(\tilde{y} - v_1)]. \quad (5)$$

The lattice action replaces in eq. (2) the derivatives by lattice derivatives, the fields without derivatives by the cell averages, and  $\int d^2x \rightarrow \sum_{\tilde{y}} V(\tilde{y})$  with cell volume  $V(\tilde{y}) = 2\Delta^2$ . The distance between neighboring cells is  $\sqrt{2}\Delta$ .

Most important for our context, the action is lattice diffeomorphism invariant. Indeed we can change the positioning of the lattice points  $x^\mu(\tilde{z})$  from the regular lattice  $x^\mu(\tilde{z}) = \Delta\tilde{z}^\mu$  to an arbitrary neighboring lattice  $x^\mu(\tilde{z}) = \Delta\tilde{z}^\mu + \xi^\mu(\tilde{z})$ . Here the cartesian coordinates  $x^\mu$  parametrize a manifold which is some region in  $\mathbb{R}^2$ . For general positions of the lattice points on the manifold the lattice derivatives read [8]

$$\begin{aligned} \partial_0 \varphi(\tilde{y}) &= \frac{1}{2V(\tilde{y})} \left\{ (x^1(\tilde{y} + v_1) - x^1(\tilde{y} - v_1)) \right. \\ &\quad \left. (\varphi(\tilde{y} + v_0) - \varphi(\tilde{y} - v_0)) \right. \\ &\quad \left. - (x^1(\tilde{y} + v_0) - x^1(\tilde{y} - v_0)) (\varphi(\tilde{y} + v_1) - \varphi(\tilde{y} - v_1)) \right\}, \\ \partial_1 \varphi(\tilde{y}) &= \frac{1}{2V(\tilde{y})} \left\{ (x^0(\tilde{y} + v_0) - x^0(\tilde{y} - v_0)) \right. \\ &\quad \left. (\varphi(\tilde{y} + v_1) - \varphi(\tilde{y} - v_1)) \right. \\ &\quad \left. - (x^0(\tilde{y} + v_1) - x^0(\tilde{y} - v_1)) (\varphi(\tilde{y} + v_0) - \varphi(\tilde{y} - v_0)) \right\}, \end{aligned} \quad (6)$$

and the cell volume becomes

$$V(\tilde{y}) = \frac{1}{2} \epsilon_{\mu\nu} (x^\mu(\tilde{y} + v_0) - x^\mu(\tilde{y} - v_0)) (x^\nu(\tilde{y} + v_1) - x^\nu(\tilde{y} - v_1)). \quad (7)$$

The cell averages (5) remain the same. The change of each term in the action due to the change of the lattice derivatives is precisely canceled by the change of the volume factor in  $\int d^2x = \sum_{\tilde{y}} V(\tilde{y})$ . The expression of the lattice action in terms of lattice derivatives and cell averages is therefore independent of the positioning of the lattice points. This crucial property is due to the particular contraction of the lattice derivatives with the  $\epsilon$ -tensor. It guarantees the standard diffeomorphism symmetry of the quantum effective action in the continuum limit [8].

We will impose periodic boundary conditions on a square lattice defined by the  $E_0$  and  $E_1$  vectors. We have checked by performing calculations on a square lattice defined by the  $x_0$  and  $x_1$  axes, that the boundary conditions do not play an important role in determining the phase structure of the theory. The functional integral

$$Z = \int \mathcal{D}\varphi e^{-S} \quad (8)$$

involves for every lattice point  $\tilde{z}$  the standard  $SO(4)$  invariant measure on the sphere  $S^3$  in the field space. The functional integral is finite for a finite number of lattice points. Our model is therefore mathematically well defined and a candidate for regularized quantum gravity.

We work with a fixed positioning of the lattice points on the regular quadratic lattice as described above. For general relativity, this corresponds to a fixed choice of coordinates.

We obtain the numerical results presented in this paper by Monte Carlo simulations using the standard Metropolis

algorithm on  $64^2$  or  $256^2$  lattices. We usually start the thermalization process in the stripe phase (see sect. V). We have checked that the system eventually ends up in the stripe phase even if we start the thermalization in the disordered phase (for the lattice sizes employed here). In this case the thermalization typically takes much longer, as the system first breaks up into domains with different orientations of stripes, which slowly equilibrate into one coherent domain.

### III. ORDER PARAMETERS AND METRICS

The continuum action (2) is invariant under translations and rotations, as well as with respect to a parity-type discrete transformation  $P: x^1 \rightarrow -x^1, \varphi_1 \leftrightarrow \varphi_2$  or time reversal  $T: x^0 \rightarrow -x^0, \varphi_1 \leftrightarrow \varphi_2$ . Similarly,  $S$  does not change under diagonal reflections  $D_\pm: x^0 \leftrightarrow \pm x^1, \varphi_1 \leftrightarrow \varphi_2$ . Another discrete symmetry is charge conjugation which is realized by complex conjugation,  $C: \varphi_i \rightarrow \varphi_i^*$ . Furthermore, the action conserves a global continuous flavor symmetry with abelian gauge group  $U(1) \times U(1)$ , corresponding to separate phase rotations for  $\varphi_1$  and  $\varphi_2$ . Since the action changes sign under  $\varphi_1 \leftrightarrow \varphi_2$  we can restrict the discussion to positive  $\beta$ . We also observe that  $S$  changes sign if only one of the fields is replaced by its complex conjugate, say  $\varphi_1 \rightarrow \varphi_1^*, \varphi_2 \rightarrow \varphi_2$ . For the reflections  $P, T, D_\pm$  we can therefore replace the accompanying reflection  $\varphi_1 \leftrightarrow \varphi_2$  by  $\varphi_1 \leftrightarrow \varphi_1^*$ . Combined flavor reflections, as  $\varphi_1 \rightarrow \varphi_2, \varphi_2 \rightarrow \varphi_1^*$ , leave the action invariant.

The lattice action is invariant under diagonal translations of  $\sqrt{2}\Delta$ , and therefore also under translations in the  $x^0$  or  $x^1$  directions by  $2\Delta$ . It is preserved by  $\pi/2$ -rotations and shares the same discrete symmetries as discussed before for continuous space, as well as the continuous  $U(1) \times U(1)$  flavor symmetry.

We find that for large enough  $\beta$  several of the discrete symmetries are spontaneously broken. Indeed, the characteristic configurations of the scalar fields change qualitatively at  $\beta_c \approx 4.6$ , as demonstrated in Fig. 1. For  $\beta > \beta_c$  one observes an order in stripes which disappears for  $\beta < \beta_c$ . For the “stripe phase” ( $\beta > \beta_c$ ) we can define order parameters  $s_i^\pm$  by defining “supercells” with sixteen lattice points. For  $s_i^+$  the field values  $\varphi_i$  within a given supercell are summed with phases according to the left part of Fig. 2, while for  $s_i^-$  we sum with the complex conjugate of the phase factors,

$$\begin{aligned} s_i^+ &= \frac{1}{L^2} \sum_{\tilde{z}} e^{i\eta(\tilde{z})} \varphi_i(\tilde{z}), \\ s_i^- &= \frac{1}{L^2} \sum_{\tilde{z}} e^{i\eta^*(\tilde{z})} \varphi_i(\tilde{z}). \end{aligned} \quad (9)$$

(More precisely, the phases  $\exp(i\eta(\tilde{z}))$  depend on the coordinates within the supercell, as defined by the left panel of Fig. 2, and  $L$  is the number of lattice points along one (diagonal) direction.) We will show in sect. V that there are four different classes of stripe configurations which can not be rotated into each other using the internal  $U(1) \times U(1)$

symmetry. Each of the four order parameters  $s_i^a$  with  $i = 1, 2$  and  $a = \pm$  signals the realization of one of the four equivalent classes of stripe configurations.

We show a characteristic order parameter as a function of  $\beta$  in Fig. 3. In the presence of such order the lattice translation symmetry is partially broken - only translations by  $4\sqrt{2}\Delta$  in the diagonal directions or by  $8\Delta$  in the  $x^0$  or  $x^1$  directions leave the equilibrium state invariant. Also the discrete reflection symmetries and the symmetry of  $\pi/2$ -lattice rotations are broken spontaneously by the direction of the stripes. The symmetry of the continuous flavor transformations gets broken spontaneously as well and one may expect Goldstone-type excitations. All these symmetries are preserved in the disordered phase for  $\beta < \beta_c$ . The discontinuity in the scalar order parameter visible in Fig. 3 indicates a first order phase transition. This will be confirmed by jump in other expectation values.

For  $\beta > \beta_c$  we will see in sect. V that lattice translations, reflections and rotations leave the equilibrium state invariant if they are combined with appropriate phase (or flavor) rotations. It is therefore perhaps more appropriate to associate the stripe phase with a spontaneous breaking of the  $U(1) \times U(1)$ -flavor symmetry. (We discuss compatibility with the Mermin-Wagner theorem at the end of sect. V.) Observables which are invariant under flavor rotations will be invariant under translations by  $E_0$  or  $E_1$ .

We are interested in geometry and therefore look for expectation values of observables that can play the role of a metric. In the continuum limit such observables should transform as second rank symmetric tensors. We will find several natural candidates for composite metric observables, with euclidean or Minkowski signature of the expectation values. They are typically invariant under flavor rotations.

Metric tensors can be constructed from derivatives of  $\varphi_i$ . Many possibilities exist for constructing objects that transform as a symmetric second rank tensors. In the continuum limit  $\varphi$  transforms under diffeomorphisms as a scalar and  $\partial_\mu \varphi$  as a covariant vector. The symmetric product of two vectors transforms therefore as symmetric second rank covariant tensor, which is precisely the transformation property of the metric. The lattice analogon replaces the derivatives by lattice derivatives (4) and the fields without derivatives by the cell averages (5). We discuss here four examples that are all invariant under the flavor symmetry of continuous phase rotations of  $\varphi_1$  and  $\varphi_2$ .

Our first candidate for a metric reads

$$\tilde{g}_{\mu\nu}^{(2)}(\tilde{y}) = 2\Delta^2 \text{Re} \left( \sum_i \partial_\mu \varphi_i^*(\tilde{y}) \partial_\nu \varphi_i(\tilde{y}) \right), \quad (10)$$

with lattice derivatives  $\partial_\mu \varphi(\tilde{y})$  given by eq. (4). On a  $L \times L$  lattice we find that the expectation value is invariant under lattice translations,

$$g_{\mu\nu}^{(2)}(\tilde{y}) = \langle \tilde{g}_{\mu\nu}^{(2)}(\tilde{y}) \rangle = N^{(2)}(\beta) \delta_{\mu\nu}, \quad (11)$$

with  $N^{(2)}(\beta)$  shown in Fig. 4. The corresponding geometry is euclidean flat space for all  $\beta$ . (The normalization factor  $N^{(2)}(\beta)$  can be absorbed by a rescaling of the metric or the

coordinates.) We observe a discontinuity of  $N^{(2)}(\beta)$  at  $\beta_c$ . The particular ground state  $g_{\mu\nu} \sim \delta_{\mu\nu}$  preserves the lattice rotations and reflections.

A second candidate is

$$\tilde{g}_{\mu\nu}^{(M)} = 8\Delta^2 \text{Re}(i\varphi_1^* \partial_\mu \varphi_1) \text{Re}(i\varphi_2^* \partial_\nu \varphi_2) + (\varphi_1 \leftrightarrow \varphi_2). \quad (12)$$

For this ‘‘Minkowski metric’’ we find that the expectation value

$$g_{\mu\nu}^{(M)} = \langle \tilde{g}_{\mu\nu}^{(M)} \rangle = N^{(M)}(\beta) \eta_{\mu\nu} \quad (13)$$

has Minkowski signature  $\eta_{\mu\nu} = \text{diag}(-1, 1)$ . Again,  $N^{(M)}(\beta)$  is plotted in Fig. 4 and shows a discontinuity at  $\beta_c$ . The expectation value (13) is the same for all lattice sites. It is invariant under (euclidean) rotations by  $\pi/2$  and under the discrete symmetries  $P$  and  $T$ .

A third metric can be written as

$$\tilde{g}_{\mu\nu}^{(MD)} = 8\Delta^2 \{ \text{Re}(i\varphi_1^* \partial_\mu \varphi_1) \text{Re}(i\varphi_1^* \partial_\nu \varphi_1) - (\varphi_1 \leftrightarrow \varphi_2) \}. \quad (14)$$

The expectation value  $g_{\mu\nu}^{(MD)} = \langle \tilde{g}_{\mu\nu}^{(MD)} \rangle$  is found as

$$g_{00}^{(MD)} = g_{11}^{(MD)} = 0, \quad g_{01}^{(MD)} = N^{(MD)}(\beta). \quad (15)$$

Again, this corresponds to a Minkowski signature since  $\det(g_{\mu\nu}^{(MD)}) = -(N^{(MD)}(\beta))^2$ . In the continuum, the metric (15) obtains from the metric (13) by a euclidean rotation of the coordinate axes by  $\pi/4$ . For the metric (15) the time- and space- axes are given by the diagonal axes in the directions of  $E_0 = (1, 1)$  and  $E_1 = (-1, 1)$ . Finally, we investigate a fourth metric

$$\tilde{g}_{\mu\nu}^{(E)} = 8\Delta^2 \text{Re}(i\varphi_1^* \partial_\mu \varphi_1) \text{Re}(i\varphi_1^* \partial_\nu \varphi_1) + (\varphi_1 \leftrightarrow \varphi_2). \quad (16)$$

The expectation value corresponds again to euclidean flat space

$$g_{\mu\nu}^{(E)} = \langle \tilde{g}_{\mu\nu}^{(E)} \rangle = N^{(E)}(\beta) \delta_{\mu\nu}. \quad (17)$$

It is remarkable that, according to the precise definition of the metric observable, one finds expectation values that correspond to flat space, either with a euclidean signature or a Minkowski signature. The signature apparently depends on the flavor structure of the metric observable. We will gain later a better understanding of this issue. The dependence of the metric observable on  $\beta$  shown in Fig. 4 conforms the discontinuity at the critical value  $\beta_c$  that signals the presence of a first order transition.

#### IV. LONG RANGE METRIC CORRELATIONS

In the stripe phase for  $\beta > \beta_c$  we observe the presence of long range correlations for fluctuations of the metric field. They decay with a power law. We define the fluctuations

$$h_{\mu\nu}^{(A)} = \tilde{g}_{\mu\nu}^{(A)} - g_{\mu\nu}^{(A)} = \tilde{g}_{\mu\nu}^{(A)} - \langle \tilde{g}_{\mu\nu}^{(A)} \rangle \quad (18)$$

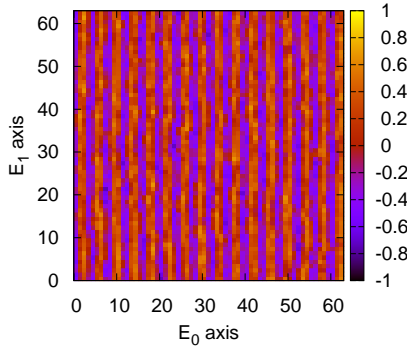
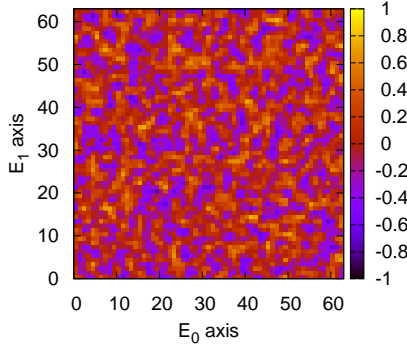


FIG. 1: The real part of  $\varphi_1$  for a typical configuration at  $\beta = 4 < \beta_c$  (top) and  $\beta = 5 > \beta_c$  (bottom) on a  $64^2$  lattice. Units along the axes are  $\sqrt{2}\Delta$ .

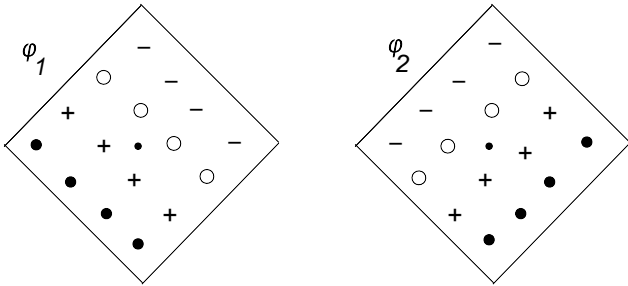


FIG. 2: Phases of fields in a supercell, 1 for +, -1 for -,  $i$  for  $\bullet$ ,  $-i$  for  $\circ$ . The small dot in the center is the location of the supercell.

for the different metric fields (10), (12), (14), (16), labeled by (A). As an example, we consider the correlation function

$$G_d^{(2)}(r) = \frac{1}{4} \left\langle (h_{00}^{(2)}(x) - h_{11}^{(2)}(x)) (h_{00}^{(2)}(y) - h_{11}^{(2)}(y)) \right\rangle \quad (19)$$

for  $r = |x - y|$ . In Fig. 5 we plot  $G_d^{(2)}(r)$  on the diagonal axis  $\sim E_0$  for three values of  $\beta$ . For  $\beta > \beta_c$  we clearly see a

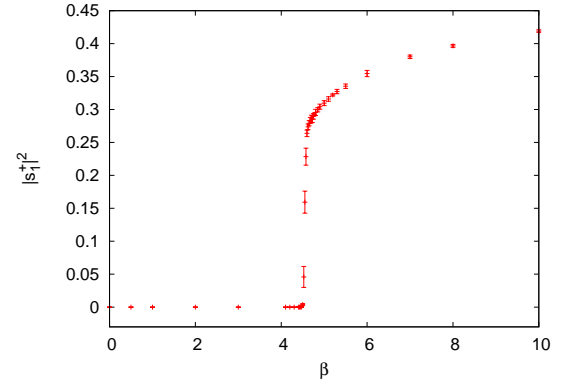


FIG. 3: The order parameter  $|s_1^+|^2$  as a function of  $\beta$ . The thermalization of the system was started from the striped phase signalled by  $s_1^+$ .

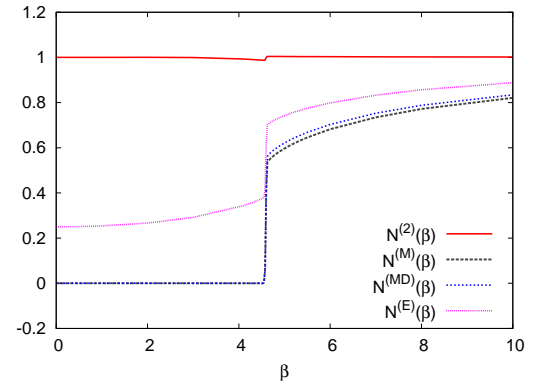


FIG. 4: Proportionality factor  $N(\beta)$  of the metric expectation values defined in eqs. (10), (12), (14) and (16), as a function of  $\beta$ .

power law decay. A fit for the range  $5\Delta \leq r \leq 50\Delta$  yields

$$G_d^{(2)}(r) = c_d r^{-\alpha_d} \quad (20)$$

with

$$\begin{aligned} \alpha_d &= 2.07 \pm 0.05, \quad c_d = 0.031 \quad \text{for } \beta = 4.7 \\ \alpha_d &= 1.99 \pm 0.03, \quad c_d = 0.0069 \quad \text{for } \beta = 20. \end{aligned} \quad (21)$$

It is remarkable that the powerlaw decay of the correlation function occurs for all values  $\beta > \beta_c$ . No tuning of a parameter, as common for critical behavior at a second phase transition, is necessary. We have excluded from the fit the points with  $r < 5\Delta$  since lattice details are relevant in this range - a possible universal behavior will only be found at distances  $r$  sufficiently large compared to the lattice distance. For  $r > 50\Delta$  the correlation becomes very small and the uncertainties large. Also finite volume effects may start to play a role.

On the other hand, we observe in the symmetric phase for  $\beta < \beta_c$  a much faster decay of the metric correlation. There are only few points before the correlation gets very

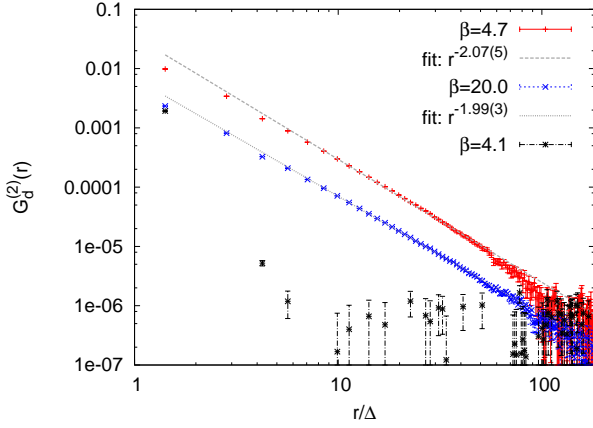


FIG. 5: Euclidean metric correlator  $G_d^{(2)}(r)$ , defined in eq. (19), as a function of distance along the  $E_0$  axis, for different couplings  $\beta$ .

small and the uncertainties large. The decay is compatible with an exponential decay or a power law with large negative exponent. Universal long distance behavior seems not to be realized in the symmetric phase, even for  $\beta$  close to  $\beta_c$ .

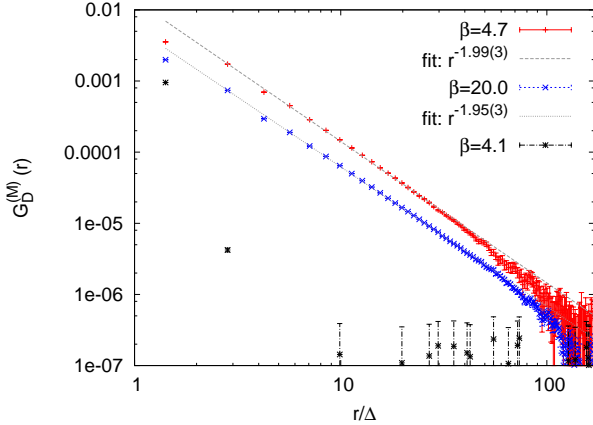


FIG. 6: Minkowski metric correlator  $G_D^{(M)}(r)$ , defined in eq. (22) as a function of distance along the  $E_0$  axis, for different couplings  $\beta$ .

The correlation function for the fluctuations  $h_{\mu\nu}^{(M)}$  of the Minkowski metric also shows a long range behavior for  $\beta > \beta_c$ . In Fig. 6 we display  $G_D^{(M)}(r)$ , defined by

$$G_D^{(M)}(r) = \frac{1}{4} \left\langle (h_{00}^{(M)}(x) + h_{11}^{(M)}(x)) (h_{00}^{(M)}(y) + h_{11}^{(M)}(y)) \right\rangle \quad (22)$$

for  $r = |x - y|$  again on the diagonal axis  $\sim E_0$ . This correlation function shows a power law decay similar to the euclidean metric  $g_{\mu\nu}^{(2)}$ , with a decay exponent close to  $\alpha = 2$ . In the following we will concentrate on the stripe phase for  $\beta > \beta_c$ . Our aim is an understanding of the nature of the

observed long range correlations for the metric observables. Further aspects of the correlation function for the metric will be discussed in sect. VIII.

## V. STRIPES

For large values of  $\beta$  the leading configurations are stripes. A characteristic stripe configuration reads

$$\varphi_1(\tilde{z}) = \frac{1}{\sqrt{2}} e^{i\alpha_1(\tilde{z})}, \quad \varphi_2(\tilde{z}) = \frac{1}{\sqrt{2}} e^{i\alpha_2(\tilde{z})}, \quad (23)$$

with phases

$$\begin{aligned} \alpha_1(\tilde{y} + (2m_1 - 1)v_0 + n_1 E_1) &= -\frac{m_1 \pi}{2}, \\ \alpha_2(\tilde{y} + (2m_2 - 1)v_1 + n_2 E_0) &= -\frac{m_2 \pi}{2}. \end{aligned} \quad (24)$$

Here  $m_i$  and  $n_i$  are arbitrary integers and  $E_0, E_1$  are given by eq. (3). For  $\varphi_1$  the stripe is in the diagonal  $E_1$ -direction, with  $\varphi_1(\tilde{z} + nE_1) = \varphi_1(\tilde{z})$ , while  $\varphi_2$  it is orthogonal to it in the diagonal  $E_0$ -direction, with  $\varphi_2(\tilde{z} + nE_0) = \varphi_2(\tilde{z})$ . A typical realistic stripe configuration with fluctuations is shown in the right part of Fig. 1. While  $\varphi_1$  is invariant under  $E_1$ -translations, it is invariant under translations in the  $E_0$ -direction only by four units,  $\varphi_1(\tilde{z} + 4nE_0) = \varphi_1(\tilde{z})$ . Similarly,  $\varphi_2$  is invariant under  $E_1$ -translations by four units,  $\varphi_2(\tilde{z} + 4nE_1) = \varphi_2(\tilde{z})$ . As a whole, the stripe configuration (23), (24) is therefore invariant under translations in the  $E_0$ - and  $E_1$ -directions by four units. This results in translation invariance in the  $x^0$ - and  $x^1$ -directions by  $8\Delta$ .

A given stripe configuration is not invariant under the continuous phase rotations of the  $U(1) \times U(1)$ -flavor symmetry. The phase changes can be used, however, to establish a symmetry of translations in the  $E_0$ - and  $E_1$ -directions by one unit combined with an appropriate phase rotation. Indeed, the combined diagonal translations,

$$t_0 : \quad \varphi'_1(\tilde{z}) = -i\varphi_1(\tilde{z} - E_0), \quad \varphi'_2(\tilde{z}) = \varphi_2(\tilde{z} - E_0), \quad (25)$$

and

$$t_1 : \quad \varphi'_1(\tilde{z}) = \varphi_1(\tilde{z} - E_1), \quad \varphi'_2(\tilde{z}) = -i\varphi_2(\tilde{z} - E_1), \quad (26)$$

leave the stripe configuration (23), (24) invariant. By virtue of these combined translations it is often sufficient to evaluate the value that a cell-observable takes for the stripe configuration only for the cell at  $\tilde{y} = 0$ . The value for the neighboring cell configuration at  $\tilde{y} = E_0$  obtains from the value of the observable for the cell  $\tilde{y} = 0$  by multiplying each factor  $\varphi_1$  by a phase factor  $-i$ . Similarly, the value in the neighboring cell at  $\tilde{y} = E_1$  is found by multiplying each factor  $\varphi_2$  by a phase  $-i$ . This can be continued for all cells. In particular, all cell-observables that are invariant under  $U(1) \times U(1)$  flavor transformations take for the stripe configuration the same value in each cell.

We first evaluate for the stripe configuration the value

of the cell action

$$\begin{aligned}\mathcal{L}(\tilde{y}) &= \epsilon^{\mu\nu} [\bar{\varphi}_1^*(\tilde{y}) \partial_\mu \varphi_1(\tilde{y}) - \bar{\varphi}_1(\tilde{y}) \partial_\mu \varphi_1^*(\tilde{y})] \\ &\quad \times [\bar{\varphi}_2^*(\tilde{y}) \partial_\nu \varphi_2(\tilde{y}) - \bar{\varphi}_2(\tilde{y}) \partial_\nu \varphi_2^*(\tilde{y})], \\ S &= 2\Delta^2 \beta \sum_{\tilde{y}} \mathcal{L}(\tilde{y}).\end{aligned}\quad (27)$$

This cell observable is independent of flavor-phase rotations and takes therefore the same values in all cells. For  $\tilde{y} = 0$  one finds for the stripe configuration (23), (24) the following values

$$\begin{aligned}\bar{\varphi}_1 &= \frac{1}{2\sqrt{2}}(1-i), \quad \bar{\varphi}_2 = \frac{1}{2\sqrt{2}}(1-i), \\ \partial_0 \varphi_1 &= -\frac{1}{2\sqrt{2}\Delta}(1+i), \quad \partial_1 \varphi_1 = -\frac{1}{2\sqrt{2}\Delta}(1+i), \\ \partial_0 \varphi_2 &= \frac{1}{2\sqrt{2}\Delta}(1+i), \quad \partial_1 \varphi_2 = -\frac{1}{2\sqrt{2}\Delta}(1+i).\end{aligned}\quad (28)$$

This yields

$$\begin{aligned}\bar{\varphi}_1^* \partial_0 \varphi_1 &= -\frac{i}{4\Delta}, \quad \bar{\varphi}_1^* \partial_1 \varphi_1 = -\frac{i}{4\Delta}, \\ \bar{\varphi}_2^* \partial_0 \varphi_2 &= \frac{i}{4\Delta}, \quad \bar{\varphi}_2^* \partial_1 \varphi_2 = -\frac{i}{4\Delta},\end{aligned}\quad (29)$$

and therefore

$$\mathcal{L}(\tilde{y}) = -\frac{1}{2\Delta^2}.\quad (30)$$

We will see below that the stripe configuration (23), (24) minimizes the action (2). (The action per lattice point takes the value  $-\beta$ .) The ground state for  $\beta \rightarrow \infty$  is therefore given by the stripe (23), (24) or one of the equivalent configurations that we will discuss below.

The metric bilinear (10) is again a cell observable that does not depend on the flavor phases and therefore takes the same value for all cells. For the stripe configuration one finds from eq. (28)

$$\tilde{g}_{00}^{(2)}(\tilde{y}) = \tilde{g}_{11}^{(2)}(\tilde{y}) = 1, \quad \tilde{g}_{01}^{(2)}(\tilde{y}) = 0.\quad (31)$$

This agrees with the asymptotic value  $N^{(2)}(\beta \rightarrow \infty) = 1$  that can be seen in Fig. 4.

We next investigate the discrete lattice reflections which leave the stripe configuration invariant. A time reflection at the axis  $\tilde{z}^0 = M$  leaves the stripe invariant if it is accompanied by  $\varphi_1 \leftrightarrow \varphi_2$  and an appropriate global phase change

$$\begin{aligned}T_M : \quad \varphi_1'(\tilde{z}^0 - M, \tilde{z}^1) &= e^{i\alpha_M} \varphi_2(-(\tilde{z}_0 - M), \tilde{z}_1), \\ \varphi_2'(\tilde{z}^0 - M, \tilde{z}_1) &= e^{-i\alpha_M} \varphi_1(-(\tilde{z}_0 - M), \tilde{z}_1),\end{aligned}\quad (32)$$

where

$$\alpha_M = -\frac{M\pi}{2}.\quad (33)$$

Similarly, a parity type invariance of the stripe configuration obtains by combining the reflection at the axis  $\tilde{z}^1 = N$  with an appropriate phase shift

$$\begin{aligned}P_N : \quad \varphi_1'(\tilde{z}^0, \tilde{z}^1 - N) &= e^{i\alpha_N} \varphi_2^*(\tilde{z}^0, -(\tilde{z}^1 - N)), \\ \varphi_2'(\tilde{z}^0, \tilde{z}^1 - N) &= e^{i\alpha_N} \varphi_1^*(\tilde{z}^0, -(\tilde{z}^1 - N)), \\ \alpha_N &= -\frac{(N+1)\pi}{2}.\end{aligned}\quad (34)$$

Since the symmetry  $P_N$  in eq. (34) also involves the complex conjugation of the fields  $\varphi_i$  it may be associated with a CP transformation. Both symmetries  $P_N$  and  $T_N$  are symmetries of the action. For a ground state preserving these symmetries an observable and its associated reflected observable must have the same expectation value.

In particular, we may consider observables that are invariant under the transformations  $\varphi_1 \leftrightarrow \varphi_2, \varphi_i \leftrightarrow \varphi_i^*$ , as well as under flavor phase rotations. Then the expectation values of observables for which only the coordinates are reflected must be the same as the ones for the original observables. The metric observable (10) is of this type. The symmetries  $T_M$  and  $P_N$  imply  $g_{01}^{(2)}(\tilde{y}) = 0$  since  $g_{01}^{(2)}$  is odd under the reflection of one coordinate.

For the diagonal reflections  $D_\pm$  we restrict the discussion to reflections at axes through the origin  $\tilde{y}_B = (0, 0)$ . Reflections at shifted axes can be obtained by a combination of those “basic reflections” with the translations  $t_0$  and  $t_1$  given by eqs. (25), (26). (Also the reflections  $T_M$  and  $P_N$  can be related in this way to suitable basis reflections at axes through the origin.) The stripe (23), (24) is indeed invariant under the diagonal reflections

$$\begin{aligned}\tilde{D}_+ : \quad \varphi_1'(\tilde{y}) &= \varphi_1(D_+ \tilde{y}), \quad \varphi_2'(\tilde{y}) = -i\varphi_2^*(D_+ \tilde{y}), \\ \tilde{D}_- : \quad \varphi_1'(\tilde{y}) &= -i\varphi_1^*(D_- \tilde{y}), \quad \varphi_2'(\tilde{y}) = \varphi_2(D_- \tilde{y}),\end{aligned}\quad (35)$$

with

$$D_+ \tilde{y} = (\tilde{y}^1, \tilde{y}^0), \quad D_- \tilde{y} = (-\tilde{y}^1, -\tilde{y}^0).\quad (36)$$

The symmetries  $\tilde{D}_+$  and  $\tilde{D}_-$  are symmetries of the action. If they are preserved by the ground state they imply  $g_{00}^{(2)}(\tilde{y}) = g_{11}^{(2)}(\tilde{y})$ . Now  $g_{01}^{(2)}$  is even under the diagonal reflections, while the difference  $d = \frac{1}{2}(\tilde{g}_{00}^{(2)} - \tilde{g}_{11}^{(2)})$  is odd.

Rotations by  $\pi/2$  can be obtained by combining reflections. For example, the combination  $T_0 \tilde{D}_+$  amounts to a rotation around the origin with an angle  $-\pi/2$ ,

$$T D_+ \tilde{y} = (\tilde{y}^1, -\tilde{y}^0).\quad (37)$$

With respect to the  $\pi/2$ -rotations both  $d$  and  $g_{01}$  are odd, while  $h = \tilde{g}_{00} + \tilde{g}_{11}$  is even.

Symmetries of the action that do not leave the stripe configuration (23), (24) invariant lead to equivalent stripes. Equivalent stripes differ from the stripe (23), (24) in position, orientation and phases, while they have the same value of the action. (If the ground state is given by one particular stripe configuration the symmetry transformations leading to equivalent, but not identical, stripes are spontaneously broken.) As an example, the action of the

stripe is not modified if  $\varphi_1$  and  $\varphi_2$  are multiplied by global phases  $e^{i\beta_1}$  and  $e^{i\beta_2}$ , respectively. Similarly, the stripes can be displaced by an arbitrary number of units  $E_0$  or  $E_1$ , or they can be rotated by  $\pi/2$ . Equivalent stripes also obtain from reflections  $P, T, D_{\pm}$  if those are accompanied by  $\varphi_1 \leftrightarrow \varphi_2$  or  $\varphi_1 \leftrightarrow \varphi_1^*$ . Pure coordinate reflections of stripes do not lead to equivalent stripes, however. For a pure coordinate reflection of the stripe (23), (24) the action becomes positive  $\mathcal{L}(\tilde{y}) = 1/(2\Delta^2)$  - such reflected stripes correspond to a maximum rather than a minimum of the action. Among the symmetries of the action that lead to equivalent (but not identical) stripes is the flavor rotation

$$R_f : \quad \varphi_1 \rightarrow \varphi_2, \quad \varphi_2 \rightarrow \varphi_1^*, \quad (38)$$

and the charge conjugation:

$$R_c : \quad \varphi_1 \rightarrow \varphi_1^*, \quad \varphi_2 \rightarrow \varphi_2^*. \quad (39)$$

They define equivalence classes of stripes that cannot be rotated into each other by phase rotations. While  $R_c$  maps  $s_i^+ \leftrightarrow s_i^-$  in eq. (9), the maps between  $s_1^{\pm}$  and  $s_2^{\pm}$  can be achieved by  $R_f$ .

For  $\beta \rightarrow \infty$  the ground state is indeed given by a particular stripe and the symmetries of flavor rotations are spontaneously broken. This issue is more delicate for finite values of  $\beta$ . Now the Mermin-Wagner theorem[10] forbids any spontaneous breaking of a continuous symmetry in two dimensions. Nevertheless, for all practical purposes the system behaves for large enough  $\beta$  as if the flavor symmetries were spontaneously broken. This issue is similar to the Kosterlitz-Thouless[11] transition where in the low temperature phase a mode with all the properties of a Goldstone boson exists. For the Kosterlitz-Thouless transition the general aspects are well understood by applying the functional renormalization group to linear and non-linear  $\sigma$ -models in two dimensions [12]. The Mermin-Wagner theorem loses its practical applicability. While it remains formally valid in the infinite volume limit, the effects of spontaneous symmetry breaking occur for a macroscopic system with arbitrary (but finite) size. We expect an analogous situation for our model, with additional complexity due to the lack of rotation or Lorentz-symmetry in the continuum limit.

The limit  $\beta \rightarrow \infty$  projects on the “ground state” for which the action takes its minimum value. For our model the ground state can be solved exactly - it is the stripe configuration (23), (24) or an equivalent stripe. In order to show this we first note that any configuration for which  $\mathcal{L}(\tilde{y})$  in eq. (27) takes its minimum value for every cell must be a minimum of the action. We next show that this is the case for the stripe.

Using in eq. (28) the definitions (4), (5), the cell action  $2\Delta^2\mathcal{L}(\tilde{y})$  can be written as a sum of terms  $\varphi_{\alpha}^*\varphi_{\beta}$  of the eight complex fields  $\varphi_{\alpha}$  in the cell. We want to show  $2\Delta^2\mathcal{L}(\tilde{y}) \geq -1$ . On everyone of the four points in the cell the condition (1) must be obeyed, leaving us with 12 angles. The flavor symmetry ensures that  $\mathcal{L}(\tilde{y})$  does not depend on the overall phases of  $\varphi_1$  and  $\varphi_2$ , such that 10 angles remain. Finding the minimum of a function of ten

angles by analytical means is rather involved. We therefore have proceeded to a numerical evaluation of  $2\Delta^2\mathcal{L}(\tilde{y})$  for random choices of the angles. For  $10^{10}$  configurations we have found no value smaller than  $-1$  within the numerical accuracy. Together with the value (30) for the stripe configuration we consider this as sufficient evidence that the stripes (23), (24) minimize the action.

## VI. ZWIBEIN

It is possible to express the action (2) in terms of a “zweibein”  $e_{\mu}^m$ . We first introduce

$$\begin{aligned} \tilde{E}_{\mu}^0 &= \sqrt{2}i\Delta(\bar{\varphi}_1^*\partial_{\mu}\varphi_1 - \bar{\varphi}_1\partial_{\mu}\varphi_1^*), \\ \tilde{E}_{\mu}^1 &= \sqrt{2}i\Delta(\bar{\varphi}_2^*\partial_{\mu}\varphi_2 - \bar{\varphi}_2\partial_{\mu}\varphi_2^*), \end{aligned} \quad (40)$$

such that

$$\begin{aligned} S &= -\frac{\beta}{2\Delta^2} \int d^2x \epsilon^{\mu\nu} \tilde{E}_{\mu}^0 \tilde{E}_{\nu}^1 \\ &= -\frac{\beta}{4\Delta^2} \int d^2x \epsilon^{\mu\nu} \epsilon_{mn} \tilde{E}_{\mu}^m \tilde{E}_{\nu}^n \\ &= -\frac{\beta}{2\Delta^2} \int d^2x \tilde{E}, \end{aligned} \quad (41)$$

with

$$\tilde{E} = \det(\tilde{E}_{\mu}^m). \quad (42)$$

In the continuum limit  $\tilde{E}_{\mu}^m$  transforms as a covariant vector. The form (41) makes the diffeomorphism symmetry of the continuum theory particularly apparent.

It is instructive to evaluate  $\tilde{E}_{\mu}^m$  for the stripe configuration (23), (24). One obtains from eq. (29)

$$\tilde{E}_{\mu}^m = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (43)$$

(With  $\tilde{E} = 1$  this reproduces  $\mathcal{L}(\tilde{y}) = -1/(2\Delta^2)$  in eq. (30).) We may bring the zweibein to a diagonal form by a suitable rotation

$$\tilde{e}_{\mu}^m = \tilde{E}_{\mu}^n R_n^m. \quad (44)$$

For  $\tilde{E}_{\mu}^m$  of the form (43) one needs

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad R^T R = 1, \quad (45)$$

such that the stripe configuration yields  $\tilde{e}_{\mu}^m = \delta_{\mu}^m$ . We will use this convention and define

$$\begin{aligned} \tilde{e}_{\mu}^0 &= \frac{1}{\sqrt{2}}(\tilde{E}_{\mu}^0 - \tilde{E}_{\mu}^1) = 2\Delta \text{Re}(i\varphi_1^*\partial_{\mu}\varphi_1) - (\varphi_1 \leftrightarrow \varphi_2) \\ \tilde{e}_{\mu}^1 &= \frac{1}{\sqrt{2}}(\tilde{E}_{\mu}^0 + \tilde{E}_{\mu}^1) = 2\Delta \text{Re}(i\varphi_1^*\partial_{\mu}\varphi_1) + (\varphi_1 \leftrightarrow \varphi_2). \end{aligned} \quad (46)$$



The action retains the form of a determinant of the zweibein

$$S = -\frac{\beta}{2\Delta^2} \int d^2x \tilde{e}, \quad \tilde{e} = \det(\tilde{e}_\mu^m). \quad (47)$$

A nonvanishing expectation value of the zweibein indicates the spontaneous breaking of symmetries of the action. This is closely related to spontaneous symmetry breaking by a given stripe configuration. We can use such symmetries in order to transform among equivalent zweibeins. For example, the flavor rotation (38) induces

$$R_f: \quad \tilde{e}_\mu^0 \rightarrow \tilde{e}_\mu^1, \quad \tilde{e}_\mu^1 \rightarrow -\tilde{e}_\mu^0. \quad (48)$$

(This amounts to a  $\pi/2$ -rotation among the flavor indices of the zweibein. Comparing with eq. (45) we note  $R_f = -R^2$ .) Similarly, charge conjugation (39) induces

$$R_c: \quad \tilde{e}_\mu^m \rightarrow -\tilde{e}_\mu^m, \quad (49)$$

such that zweibeins with opposite sign are equivalent.

As usual, the zweibein can be used to define a metric

$$\tilde{g}_{\mu\nu}^{(S)} = \hat{\eta}_{mn} \tilde{e}_\mu^m \tilde{e}_\nu^n \quad (50)$$

with  $\hat{\eta}_{mn} = \hat{\eta}_{nm}$ . We define the “euclidean metric”  $\tilde{g}_{\mu\nu}^{(E)}$  by choosing  $\hat{\eta}_{mn} = \delta_{mn}$ ,

$$\tilde{g}_{\mu\nu}^{(E)} = \tilde{e}_\mu^0 \tilde{e}_\nu^0 + \tilde{e}_\mu^1 \tilde{e}_\nu^1. \quad (51)$$

This coincides with eq. (16). For the stripe configuration (23), (24) one finds  $\tilde{g}_{\mu\nu}^{(E)} = \delta_{\mu\nu}$ . Similar to  $\tilde{g}_{\mu\nu}^{(2)}$  in eq. (10) this metric observable is independent of the flavor phases, and remains unchanged for  $\varphi_1 \leftrightarrow \varphi_2$  or  $\varphi_i \leftrightarrow \varphi_i^*$ . In contrast to eq. (10) it involves four powers of fields  $\varphi_i$ , however.

Another possibility uses  $\hat{\eta}_{mn} = \eta_{mn} = \text{diag}(-1, 1)$ . The metric corresponding to eq. (12),

$$\tilde{g}_{\mu\nu}^{(M)} = -\tilde{e}_\mu^0 \tilde{e}_\nu^0 + \tilde{e}_\mu^1 \tilde{e}_\nu^1, \quad (52)$$

takes for the stripe configuration the values  $\tilde{g}_{\mu\nu}^{(M)} = \eta_{\mu\nu}$ . For large enough  $\beta$  one therefore expects that the expectation value  $\langle g_{\mu\nu}^{(M)} \rangle$  has a Minkowski signature,  $\langle g_{\mu\nu}^{(M)} \rangle = N^{(M)}(\beta) \eta_{\mu\nu}$ . We observe that  $g_{\mu\nu}^{(M)}$  is again invariant with respect to phase changes of  $\varphi_1$  or  $\varphi_2$  and to the interchange  $\varphi_1 \leftrightarrow \varphi_2$ . It changes sign, however, under the exchange  $\varphi_1 \leftrightarrow \varphi_1^*$ . In consequence, the diagonal reflections  $\tilde{D}_\pm$  change  $g_{00}^{(M)} \leftrightarrow -g_{11}^{(M)}$  while  $g_{01}^{(M)}$  is odd. Thus  $\eta^{\mu\nu} g_{\mu\nu}^{(M)} = -g_{00}^{(M)} + g_{11}^{(M)}$  is even and  $g_{00}^{(M)} + g_{11}^{(M)}$  odd. This additional minus sign extends to  $\pi/2$ -rotations under which  $-g_{00}^{(M)} + g_{11}^{(M)}$  and  $g_{01}^{(M)}$  are even and  $g_{00}^{(M)} + g_{11}^{(M)}$  is odd. The expectation values of the metric (12) or (52) singles out a time direction. This direction depends on the stripe orientation. The transformation (48) switches between time and space directions. Thus the difference between space and time occurs as an effect of spontaneous symmetry breaking.

We have evaluated the expectation value of the zweibein and find in the striped phase for  $\beta > \beta_c$ ,

$$e_\mu^m = \langle \tilde{e}_\mu^m \rangle = N^{(e)}(\beta) \delta_\mu^m. \quad (53)$$

This holds if the striped phase is characterized by the configuration (23), (24). The expectation values in the equivalent striped phases are given by transformations similar to eqs. (48) and (49). As usual, a metric can be defined as a quadratic expression in the zweibein

$$g_{\mu\nu}^{(Me)} = e_\mu^m e_\nu^n \eta_{mn} = ((N^{(e)}(\beta))^2) \eta_{\mu\nu}. \quad (54)$$

It differs from the Minkowski metric  $\langle \tilde{g}_{\mu\nu}^{(M)} \rangle$ , cf. eq. (52), by the contribution of local zweibein fluctuations. In Fig. 7 we display  $(N^{(e)}(\beta))^2$  together with  $N^{(M)}(\beta)$ . The difference is small and we can approximate  $g_{\mu\nu}^{(M)}$  by  $g_{\mu\nu}^{(Me)}$  with good accuracy.

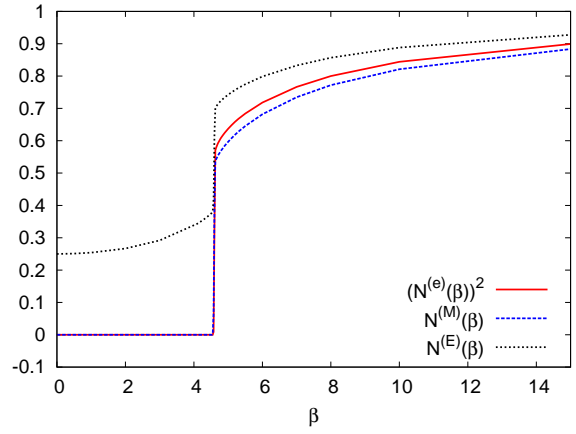


FIG. 7: Expectation values of the zweibein and metrics, as expressed by the proportionality factors  $(N^{(e)})^2$ ,  $N^{(M)}$  and  $N^{(E)}$ , as a function of  $\beta$ .

We can go one step further, and try to see if the zweibein expectation value is well described by the disconnected part of the scalar correlator, that is

$$\begin{aligned} e^{(\varphi)}_\mu{}^0 &= 2\Delta \text{Re}(i \langle \varphi_1^* \rangle \langle \partial_\mu \varphi_1 \rangle) - (\varphi_1 \leftrightarrow \varphi_2), \\ e^{(\varphi)}_\mu{}^1 &= 2\Delta \text{Re}(i \langle \varphi_1^* \rangle \langle \partial_\mu \varphi_1 \rangle) + (\varphi_1 \leftrightarrow \varphi_2). \end{aligned} \quad (55)$$

Here  $\langle \varphi_i \rangle \equiv \langle \bar{\varphi}(\tilde{y}) \rangle$  denote the averages of the cell averaged scalar fields, which are space dependent in the broken phase, and  $\langle \partial_\mu \varphi_i(\tilde{y}) \rangle$  are averages of lattice derivatives.

This quantity has the same structure as  $e_\mu^m$ ,

$$e^{(\varphi)}_\mu{}^m = N^{(e\varphi)}(\beta) \delta_\mu^m. \quad (56)$$

In Fig. 8 we compare  $N^{(e)}(\beta)$  to  $N^{(e\varphi)}(\beta)$ , and note that the disconnected zweibein describes the full zweibein to a reasonable approximation, and the quality of the agreement gets better at larger  $\beta$ .

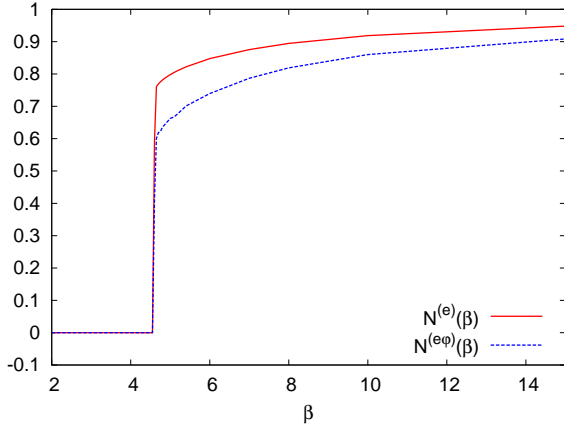


FIG. 8: Proportionality factors  $N^{(e)}(\beta)$  and  $N^{(e\varphi)}(\beta)$ , defined in eqs. (53) and (56), as a function of  $\beta$ .

In the continuum limit the presence of order parameters  $e_\mu^m$  singles out particular directions since both  $e_\mu^0$  and  $e_\mu^1$  transform as a vector. A rotation of the axes away from the  $x^0$ - and  $x^1$ -direction does not leave the zweibein diagonal, such that the diagonal form (53) singles out a particular coordinate frame. The same holds for Lorentz-transformations. In the usual vielbein formulation of geometry [9] the lack of rotation - or Lorentz- symmetry for any given value of the vielbein is compensated by an accompanying transformation of the Lorentz frame. A simultaneous rotation between the vectors  $e_\mu^0$  and  $e_\mu^1$ , together with a coordinate rotation, leaves the “ground state vielbein” of the type (53) invariant. This guarantees rotation symmetry for flat euclidean space. A similar property with respect to Lorentz symmetry ensures Lorentz symmetry for Minkowski space.

In the Cartan-formulation the euclidean metric  $g_{\mu\nu} = e_\mu^m e_\nu^n \delta_{mn}$  is obtained from the vielbein by contraction with the invariant tensor  $\delta_{mn}$ , such that the metric is rotation invariant. Then the metric correlation functions transform under rotations covariantly as dictated by their tensor properties, provided that there is no other source of rotation symmetry breaking. Similar statements hold for the Lorentz covariance of the metric  $g_{\mu\nu}^{(Me)}$  (54) and its correlation functions since  $\eta_{mn}$  is a Lorentz invariant.

The invariance of the zweibein under combined space and internal rotations is, however, not sufficient in order to guarantee a rotation invariant setting. In our model the expectation value of scalar fields in the form of stripes breaks the rotation symmetry without the possibility of an internal compensating transformation. As a result, a violation of rotation symmetry for the metric correlation functions in the continuum becomes plausible, and we discuss this issue in sect. VIII.

Furthermore, in our setting the existence of appropriate internal rotation or Lorentz transformations among the zweibein components  $e_\mu^0$  and  $e_\mu^1$  is not guaranteed a priori. We investigate this question in the next section. There we find that no transformation of the field variables can

account for a euclidean rotation among  $e_\mu^0$  and  $e_\mu^1$ . In consequence, we expect for the continuum limit the presence of “preferred axes” and a violation of the continuous rotation symmetry. The issue of Lorentz symmetry is more subtle and will be discussed in the next section.

## VII. GENERALIZED LORENTZ TRANSFORMATIONS

The formulation of the action (47) in terms of the zweibein suggests an investigation if a continuous Lorentz type symmetry could be present. Such a symmetry would have to act among the different components of the scalar field. Indeed, if we can find a transformation of the scalars such that  $(\tilde{e}_\mu^0, \tilde{e}_\mu^1)$  transform as a two-component vector (with index  $m$ ), then the contraction with  $\epsilon_{mn}$  in eq. (41) yields an invariant. This holds for “generalized Lorentz-transformations” corresponding to the groups  $SO(2)$  or  $SO(1,1)$ . (We employ here the name of “generalized Lorentz transformation” since they act on the zweibein in analogy to the usual Lorentz transformations. No fermions or spinor representations are involved, however, in our setting.)

Expressing the two complex scalars  $\varphi_i$  in terms of four real components  $\psi_\alpha$ ,

$$\varphi_1 = \psi_1 + i\psi_3, \quad \varphi_2 = \psi_2 + i\psi_4, \quad \sum_{\alpha=1}^4 \psi_\alpha^2 = 1, \quad (57)$$

we can write the zweibein as

$$\tilde{e}_\mu^m = 2\Delta\psi_\alpha(\sigma^m)_{\alpha\beta}\partial_\mu\psi_\beta, \quad (58)$$

with real antisymmetric  $4 \times 4$  matrices

$$\sigma^0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (59)$$

Under infinitesimal transformations

$$\delta\psi = \alpha A\psi \quad (60)$$

the zweibein transforms as

$$\delta e_\mu^m = 2\Delta\alpha\psi(\sigma^m A + A^T\sigma^m)\partial_\mu\psi. \quad (61)$$

A realization of the Lorentz group  $SO(1,1)$  requires

$$\sigma^0 A + A^T \sigma^0 = \sigma^1, \quad \sigma^1 A + A^T \sigma^1 = \sigma^0, \quad (62)$$

such that

$$\delta\tilde{e}_\mu^0 = \alpha\tilde{e}_\mu^1, \quad \delta\tilde{e}_\mu^1 = \alpha\tilde{e}_\mu^0. \quad (63)$$

The most general solution of eq. (62) is

$$A = \begin{pmatrix} \beta + (1 - \gamma)\tau_3 & \epsilon + \eta\tau_3 \\ \zeta + \chi\tau_3 & -\beta + \gamma\tau_3 \end{pmatrix}, \quad (64)$$

with arbitrary coefficients  $\beta, \gamma, \epsilon, \eta, \zeta, \chi$  and Pauli matrices  $\tau_k$ . On the other hand, a realization of  $SO(2)$  would replace the first equation (62) by  $\sigma^0 A + A^T \sigma^0 = -\sigma^1$ , while keeping the second one unchanged. No solution for  $A$  exists in this case. At this stage an internal Lorentz transformation remains possible, while internal euclidean rotations cannot be realized for our setting of scalar fields.

For no choice of parameters  $\beta \dots \chi$  the matrix  $A$  in eq. (64) is antisymmetric. The infinitesimal transformation (60) does therefore not respect the condition  $\psi_\alpha \psi_\alpha = 1$  for all allowed values  $\psi_\alpha$ . It is not a genuine symmetry of our model. The basic reason is that  $SO(1, 1)$  is a noncompact group, while any constraint on the  $\psi_\alpha$  that leads to a compact manifold (as the sphere  $S^3$  in our case) admits no noncompact isometries. The issue may be understood in more detail by specializing to  $A = \frac{1}{2} \text{diag}(1, -1, 1, -1)$  or  $\gamma = 1/2, \beta = \epsilon = \eta = \zeta = \chi = 0$ . The transformation (60) induces a change in the length of the vector  $\psi_\alpha$ ,

$$\delta(\psi_\alpha \psi_\alpha) = \alpha(\psi_1^2 - \psi_2^2 + \psi_3^2 - \psi_4^2). \quad (65)$$

This does not vanish for arbitrary configurations which obey  $\psi_\alpha \psi_\alpha = 1$ . Nevertheless, it vanishes for the subclass of configurations which obey  $\psi_1^2 + \psi_3^2 = \psi_2^2 + \psi_4^2$  or  $|\varphi_1|^2 = |\varphi_2|^2$ . For example, this condition is obeyed for the stripe configuration (23), (24).

This observation has an interesting consequence. For any stripe configuration (23), (24) we can infinitesimally increase all  $|\varphi_1|$  and decrease all  $|\varphi_2|$ , such that  $|\varphi_1|^2 + |\varphi_2|^2$  remains unity for all lattice points. Keeping the phases (24) fixed this does not change the action. Such a transformation amounts to an infinitesimal Lorentz rotation among the zweibein components  $e_\mu^m$  according to eq. (63).

We summarize that the action (2) is invariant under Lorentz transformations of the group  $SO(1, 1)$

$$\delta\varphi_1(x) = \frac{1}{2}\alpha(x)\varphi_1(x), \quad \delta\varphi_2(x) = -\frac{1}{2}\alpha(x)\varphi_2(x). \quad (66)$$

For the continuum action this symmetry is even a local symmetry, with transformation parameter  $\alpha(x)$  depending arbitrarily on the position  $x$ . Indeed, the inhomogeneous part  $\sim \partial_\mu \alpha$  resulting from the derivatives cancels. The local character of the Lorentz symmetry is, however, not respected by the lattice regularization - the lattice action is invariant only for constant  $\alpha$ . Furthermore, the non-linear constraint (1) is not compatible with the Lorentz symmetry.

### VIII. METRIC CORRELATION FUNCTIONS AND SYMMETRIES

In this section we investigate the behaviour of fluctuations of the metric around the expectation value,

$$h_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}. \quad (67)$$

The correlation function,

$$G_{\mu\nu,\rho\sigma}(x, y) = \langle h_{\mu\nu}(x) h_{\rho\sigma}(y) \rangle, \quad (68)$$

characterizes the response of the metric to a source (energy momentum tensor) in linear order. In our model we find in the metric sector a non-trivial scaling behavior without a finite correlation length. In Fig. 9 we exhibit the two

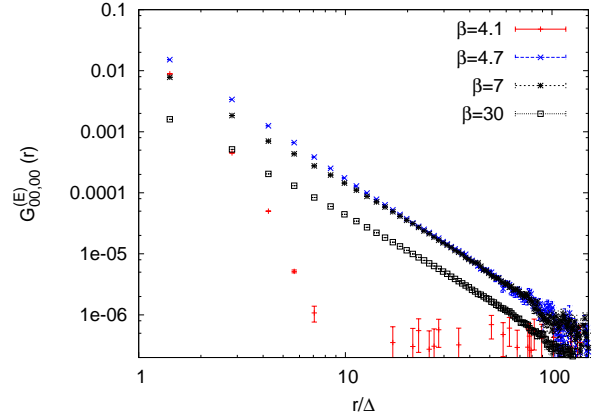


FIG. 9: Euclidean metric correlator  $G_{00,00}^{(E)}$  for different values of  $\beta$ , parallel to the  $E_0$  axis.

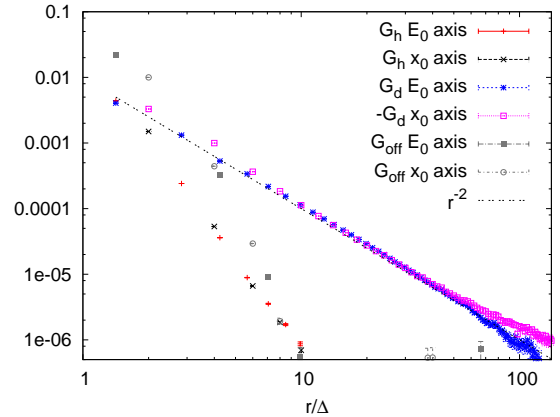


FIG. 10: Diagonal euclidean metric correlators, defined in eq. (69), at  $\beta = 10$ .

point function  $G_{00,00}$  for the euclidean metric (16). We observe a power law decay with the exponent  $\alpha = 2$  at larger couplings, while close to  $\beta_c$  the power is slightly bigger.

To disentangle this behavior we must decompose this correlator with respect to the discrete symmetries of the system. We list in Table I the behavior of the various metric components with respect to the discrete lattice symmetries, with + for even and - for odd. Here the eigenstates of the reflection symmetries are denoted by  $g_{01}$ ,  $h = h_{00} + h_{11}$  and  $d = \tilde{h}_{00} = h_{00} - \frac{1}{2}h\delta_{00} = \frac{1}{2}(h_{00} - h_{11})$ . Correlation functions involving two powers of a given eigenstate are even with respect to all discrete symmetries. As result, the correlations

$(E)$	$\frac{\pi}{2}$ -rotations	$P, T$	$D_{\pm}$
$h = h_{00} + h_{11}$	+	+	+
$d = \frac{1}{2}(h_{00} - h_{11})$	-	+	-
$h_{01}$	-	-	+

TABLE I: Symmetry properties of the euclidean metric with respect to the discrete symmetries of the lattice.

$$\begin{aligned}
G_h(x-y) &= \langle h(x)h(y) \rangle \\
&= G_{00,00} + G_{00,11} + G_{11,00} + G_{11,11}, \\
G_d(x-y) &= \langle d(x)d(y) \rangle \\
&= \frac{1}{4}(G_{00,00} - G_{00,11} - G_{11,00} + G_{11,11}), \\
G_{off}(x-y) &= \langle h_{01}(x)h_{01}(y) \rangle = G_{01,01},
\end{aligned} \tag{69}$$

are invariant if the difference  $y - x$  is rotated by an angle  $\pi/2$ . We plot these correlation functions a function of distance in Fig. 10. We observe that the decay of  $G_d$  for large  $(x - y)$  is reasonably fitted by a power law

$$G_d(x-y) = A_d |x-y|^{-\alpha} \tag{70}$$

with  $\alpha = 2$ , while  $G_h$  and  $G_{off}$  show a short distance behaviour with a steeper decay. The non-diagonal correlation functions of the euclidean metric described below show also a short distance behaviour. The correlator  $G_{00,00}(x, y)$  is thus a linear combination of the correlators above, some of which have short distance behaviour, and some has a long distance behaviour. For  $\beta$  close to  $\beta_c$  the short distance contributions change the short distance behaviour slightly, while for larger  $\beta$  the correlator is dominated by  $G_d$ , which has a decay exponent independent of  $\beta$ .

In the continuum limit for  $r \gg \Delta$  the correlator  $G_d$  is the only non-vanishing correlation function on the  $x_0$  and  $E_0$  axes for the euclidean metric  $g_{\mu\nu}^{(E)}$ . The tensor properties of the metric imply that for a  $\pi/4$ -rotation  $G_d$  should be replaced by  $G_{off}$  if rotation symmetry is realized. This is clearly not the case, therefore the euclidean rotation symmetry is broken in the continuum limit. The behavior of the correlation functions is related to an expansion of the effective action in second order in the fields. Our findings clearly indicate that the effective action is not of the simple form discussed in the Appendix A.

Using the discrete symmetries above one can find that certain correlation functions have to vanish, as described below. For example, the correlation function

$$\langle d(x)h_{01}(y) \rangle = \frac{1}{2}(G_{0001} - G_{1101}). \tag{71}$$

is even under  $\pi/2$ -rotations. Since this correlation is odd under  $P, T$  and  $D_{\pm}$  it has to vanish on the  $x^0$ - and  $x^1$ -axes, as well as on the diagonal axes  $\sim E_0$  or  $\sim E_1$ . On the other hand, the correlations

$$G_{hd}(x-y) = \langle h(x)d(y) \rangle = G_{00,00} - G_{11,11} \tag{72}$$

$(M)$	$\frac{\pi}{2}$ -rotations	$P, T$	$D_{\pm}$
$H_M = -h_{00}^{(M)} + h_{11}^{(M)}$	+	+	+
$D_M = -\frac{1}{2}(h_{00}^{(M)} + h_{11}^{(M)})$	-	+	-
$h_{01}^{(M)}$	+	-	-

TABLE II: Symmetry properties of the Minkowski metric with respect to the discrete symmetries of the lattice.

and

$$G_{hoff}(x-y) = \langle h(x)g_{01}(y) \rangle = G_{00,01} + G_{11,01} \tag{73}$$

change sign if  $y - x$  is rotated by  $\pi/2$ . The correlation  $G_{hd}$  vanishes if  $y - x$  is parallel to one of the diagonal axes  $\sim E_0$  or  $\sim E_1$ , while  $G_{hoff}$  is zero if  $y - x$  is parallel to the  $x^0$ - or  $x^1$ -axis. These features are confirmed by our numerical results.

Equivalently, we can investigate the action of the discrete symmetries directly for the correlations (68). For example, the correlation functions of the type

$$G_{\mu\nu 01}(\tilde{y}) = \langle h_{\mu\nu}(0)h_{01}(\tilde{y}) \rangle \tag{74}$$

have to obey

$$\begin{aligned}
G_{0001}(T\tilde{y}) &= G_{0001}(P\tilde{y}) = -G_{0001}(\tilde{y}), \\
G_{1101}(T\tilde{y}) &= G_{1101}(P\tilde{y}) = -G_{1101}(\tilde{y}), \\
G_{0101}(T\tilde{y}) &= G_{0101}(P\tilde{y}) = G_{0101}(\tilde{y}),
\end{aligned} \tag{75}$$

for

$$\tilde{y} = (\tilde{y}^0, \tilde{y}^1), \quad T\tilde{y} = (-\tilde{y}^0, \tilde{y}^1), \quad P\tilde{y} = (\tilde{y}^0, -\tilde{y}^1). \tag{76}$$

In particular, the correlations  $G_{0001}$  and  $G_{1101}$  have to vanish on the  $x^0$ - and  $x^1$ -axes, i.e. for  $\tilde{y} = (\tilde{y}^0, 0)$  or  $\tilde{y} = (0, \tilde{y}^1)$ . For  $\tilde{y}$  on one of the diagonal axes, e.g. for  $\tilde{y} = (m, m)$  or  $\tilde{y} = (m, -m)$  this implies for the correlations

$$\begin{aligned}
\langle h_{00}(0)d(\tilde{y}) \rangle &= -\langle h_{11}(0)d(\tilde{y}) \rangle, \\
\langle h_{01}(0)d(\tilde{y}) \rangle &= 0.
\end{aligned} \tag{77}$$

The symmetry arguments are the same for all euclidean metrics - they hold both for  $h_{\mu\nu}^{(2)}$  and  $h_{\mu\nu}^{(E)}$ .

The correlation functions for the Minkowski metric behave similarly to the euclidean correlations. The decomposition is now done with respect to a Minkowski signature, as displayed in Table II. In Fig. 11 we show the diagonal correlations

$$\begin{aligned}
G_H(x-y) &= \langle H_M(x)H_M(y) \rangle \\
&= G_{00,00}^{(M)} - G_{00,11}^{(M)} - G_{11,00}^{(M)} + G_{11,11}^{(M)}, \\
G_D(x-y) &= \langle D_M(x)D_M(y) \rangle \\
&= \frac{1}{4}(G_{00,00}^{(M)} + G_{00,11}^{(M)} + G_{11,00}^{(M)} + G_{11,11}^{(M)}), \\
G_{off}(x-y) &= \langle h_{01}^{(M)}(x)h_{01}^{(M)}(y) \rangle = G_{01,01}^{(M)}.
\end{aligned} \tag{78}$$

Similarly to the euclidean case,  $G_D$  shows a long distance behaviour of a power law decay with  $\alpha \approx 2$ . Along with  $G_{D,off}$  this is the dominant correlation function in the continuum limit. The correlation function  $G_{off}$  for the off diagonal metric components also shows a power law decay and remains present in the continuum limit, even though its value is smaller the  $G_D$  and  $G_{D,off}$  by an order of magnitude. On the other hand,  $G_H$  decays quickly and plays no role in the continuum limit. In Fig. 12 we show the correlator  $G_{00,00}^{(M)}$  which behaves similarly to  $G_{00,00}^{(E)}$  shown in Fig. 9.

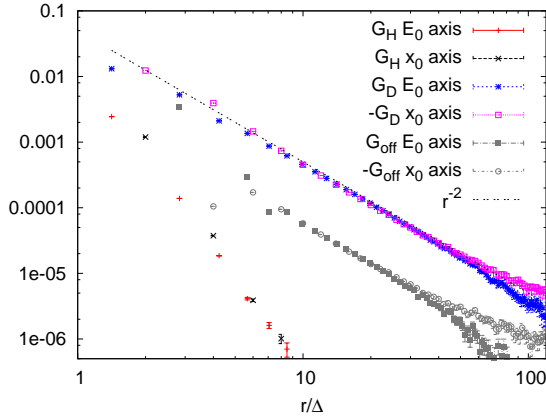


FIG. 11: Diagonal Minkowski metric correlators, defined in eq. (78), for  $\beta = 10$ .

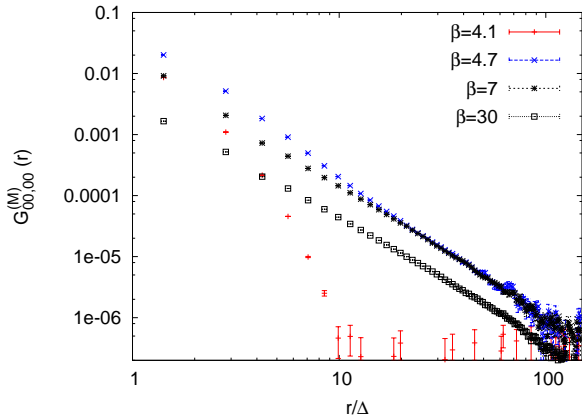


FIG. 12: Minkowski metric correlator  $G_{00,00}^{(M)}$  for different values of  $\beta$ , parallel to the  $E_0$  axis.

## IX. ZWEIBEIN CORRELATIONS

The intriguing features of the correlations for the euclidean and Minkowski metric can be understood better in terms of the correlation functions for the zweibein fluctuations

$$f_\mu^m = \tilde{e}_\mu^m - e_\mu^m = \tilde{e}_\mu^m - \langle \tilde{e}_\mu^m \rangle. \quad (79)$$

$(Z)$	$\frac{\pi}{2}$ -rotations	$P, T$	$D_\pm$
$s$	+	+	+
$a$	-	+	-
$u$	-	-	+
$v$	+	-	-

TABLE III: Symmetry properties of zweibein fluctuations.

As we have discussed above the expectation value  $e_\mu^m$  preserves the discrete symmetries  $P, T, D_\pm$  and  $\pi/2$ -rotations. We classify the zweibein fluctuations according to these symmetries

$$\begin{aligned} s &= \frac{1}{2}(f_0^0 + f_1^1), \quad a = \frac{1}{2}(f_0^0 - f_1^1), \\ u &= \frac{1}{2}(f_0^1 + f_1^0), \quad v = \frac{1}{2}(f_0^1 - f_1^0). \end{aligned} \quad (80)$$

The transformation properties of the eigenstates  $s, a, u, v$  under  $P, T, D_\pm$  and  $\pi/2$ -rotations are collected in table III.

This implies that the correlations  $\langle su \rangle, \langle sv \rangle, \langle au \rangle$  and  $\langle av \rangle$  must vanish on the  $x^0$ - and  $x^1$ - axes, while  $\langle sa \rangle, \langle sv \rangle, \langle au \rangle, \langle uv \rangle$  are zero on the diagonal axes  $\sim E_0$  or  $\sim E_1$ . A rotation of the axes by  $\pi/2$  leads to a minus sign for the correlations  $\langle sa \rangle, \langle su \rangle, \langle av \rangle$  and  $\langle uv \rangle$ , while the other combinations are invariant.

The zweibein correlators also show a powerlaw decay similarly to the metric correlators, as shown in Fig. 13. We also plot the non-vanishing non-diagonal correlations on the  $E_0$  and  $x_0$  axes in Fig. 14. Again, only some modes exhibit the slow, power law decay, while other modes show faster decay, similar to the correlations of the metric tensor. The correlations relevant for the continuum limit are  $\langle aa \rangle, \langle vv \rangle$  and  $\langle av \rangle$ .

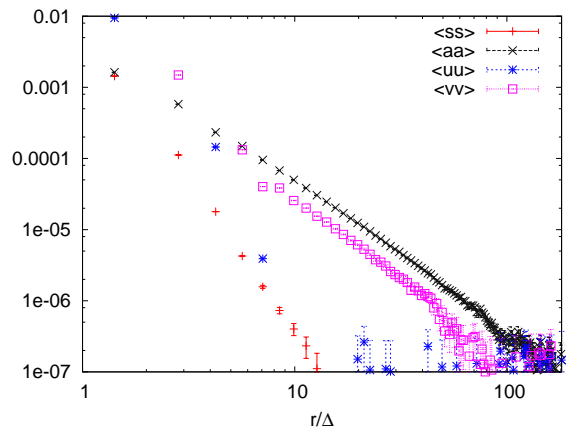


FIG. 13: Diagonal correlations of zweibein fluctuations on the  $E_0$  axis measured at  $\beta = 7$ .

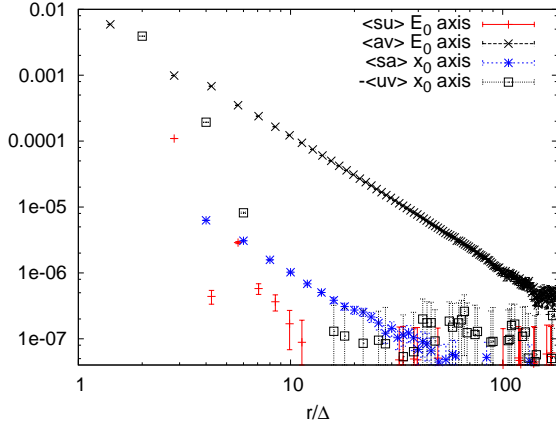


FIG. 14: Non-vanishing non-diagonal correlations of zweibein fluctuations on the  $E_0$  and  $x_0$  axes measured at  $\beta = 7$ .

The fluctuations of the metrics (50) are related to the zweibein fluctuations by

$$\begin{aligned} h_{\mu\nu}^{(S)} &= \tilde{e}_\mu^m \tilde{e}_\nu^n \hat{\eta}_{mn} - \langle \tilde{e}_\mu^m \tilde{e}_\nu^n \rangle \hat{\eta}_{mn} \\ &= \{f_\mu^m f_\nu^n - e_\mu^m f_\nu^n - f_\mu^m e_\nu^n + e_\mu^m e_\nu^n \\ &\quad - \tilde{e}_\mu^m \tilde{e}_\nu^n\} \hat{\eta}_{mn}. \end{aligned} \quad (81)$$

Taking the expectation value of eq. (81) and employing  $\langle h_{\mu\nu}^{(S)} \rangle = 0$ ,  $\langle f_\mu^m \rangle = 0$ , relates the difference between the metrics (13) and (54) to the local zweibein fluctuations. This holds both for euclidean and Minkowski signature, such that for  $S = M, E$  one finds

$$g_{\mu\nu}^{(S)} - g_{\mu\nu}^{(Se)} = \langle f_\mu^m f_\nu^n \rangle \hat{\eta}_{mn}. \quad (82)$$

Furthermore, if we neglect higher correlations  $\sim \langle f^3 \rangle, \langle f^4 \rangle_c$  one obtains an approximate relation between the metric correlations for  $g_{\mu\nu}^{(S)}$  and the zweibein correlations. (Here  $\langle f^4 \rangle_c \sim \langle (f^2(x) - \langle f^2(x) \rangle)(f^2(y) - \langle f^2(y) \rangle) \rangle$ , with indices and  $\hat{\eta}$  omitted.) This leading order relation reads

$$\begin{aligned} \langle h_{\mu\nu}^{(S)}(x) h_{\rho\sigma}^{(S)}(y) \rangle &\approx F_{\mu\rho}^{mr} e_{\nu m} e_{\sigma r} \\ &\quad + F_{\mu\sigma}^{ms} e_{\nu m} e_{\rho s} + F_{\nu\rho}^{nr} e_{\mu n} e_{\sigma r} + F_{\nu\sigma}^{ns} e_{\mu n} e_{\rho s}, \end{aligned} \quad (83)$$

where we define

$$F_{\mu\nu}^{mn} = \langle f_\mu^m(x) f_\nu^n(y) \rangle \quad (84)$$

and

$$e_{\mu n} = e_\mu^m \hat{\eta}_{mn} = N^{(e)}(\beta) \hat{\eta}_{\mu n}. \quad (85)$$

For the euclidean metric correlation this yields in the striped phase given by eq. (23)

$$\begin{aligned} G_{00,00}^{(E)} &= 4F_{00}^{00}, \quad G_{11,11}^{(E)} = 4F_{11}^{11}, \\ G_{00,11}^{(E)} &= G_{11,00}^{(E)} = 4F_{01}^{01} = 4F_{11}^{00}, \\ G_{01,01}^{(E)} &= F_{11}^{00} + 2F_{10}^{01} + F_{00}^{11}. \end{aligned} \quad (86)$$

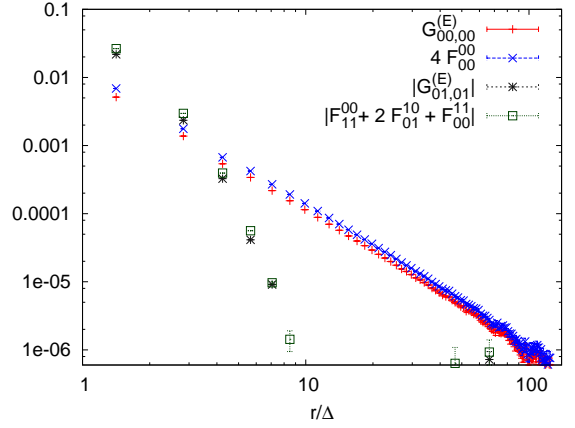


FIG. 15: Correlations of the euclidean metric compared with estimates from zweibein correlations according to eq. (86) measured at  $\beta = 10$ . Since  $G_{01,01}^{(E)}$  has an alternating sign, we show the absolute value.

In Fig. 15 we show that the metric correlations are well approximated using the estimate (86) in terms of the zweibein correlators. This is true even for the case of a fastly decaying mode. This evidence means that the higher order zweibein correlations quite small. It also implies that the power-law behaviour of the metric correlations is governed by the power law behaviour of the zweibein correlators. In Fig. 16 we show the  $\langle aa \rangle$  correlator for different values of  $\beta$ . Within our precision the decay exponent  $\alpha \approx 2$  is found to be independent of  $\beta$  in the stripe phase for  $\beta > \beta_c$ .

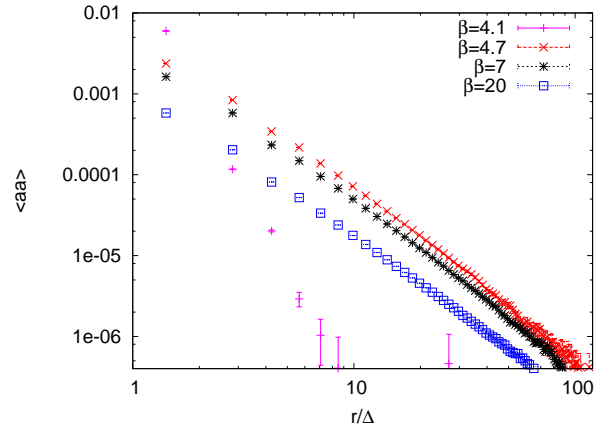


FIG. 16: Correlations  $\langle aa \rangle$  of zweibein fluctuations on the  $E_0$  axis for different values of  $\beta$ .

Finally, the scalar correlations are qualitatively different from the metric and zweibein correlators. Some typical scalar correlators are shown in Fig. 17. As one observes, they can be approximated by an exponential decay rather than a power law. The decay constant is roughly independent of the coupling above  $\beta_c$ , and the overall amplitude



of the correlations decays with increasing  $\beta$ . For the scalar correlator we must take into account that the ground state in the broken phase is only invariant with respect to translations by  $4\sqrt{2}\Delta$ . Therefore we plot the scalar correlator for distances which are a multiple of  $4\sqrt{2}\Delta$ .

We have shown in Fig. 7 that the metric expectation value is well approximated by disconnected contributions, i.e. the zweibein expectation values. In the first part of this section we have shown that also the metric correlators are well described by the zweibein correlators, neglecting higher order zweibein correlators. In this sense the metric correlations are largely determined by the zweibein. The relation of the zweibein and the scalars seem to be different. The zweibein expectation value is well described by the disconnected part, i.e. the scalar expectation values, as shown in Fig. 8. One may ask if this can be extended to the correlation functions by neglecting higher order connected correlation functions of the scalars. We address this issue in appendix C. However, the different qualitative behavior of zweibein correlators and scalar correlators make it unlikely that the zweibein correlator can be approximated by a linear combination of the scalar correlators.

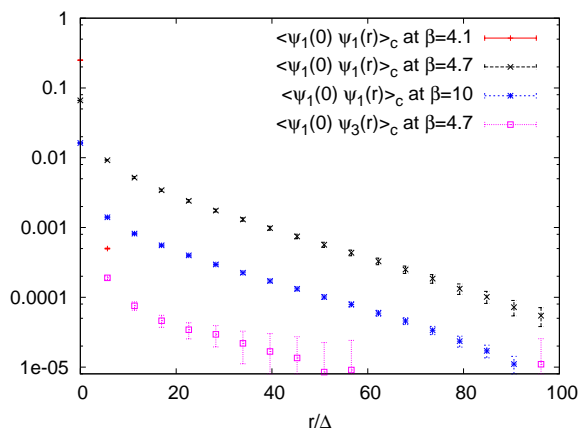


FIG. 17: Typical scalar correlators along the  $E_0$  axis measured at  $\beta = 4.9$ . Note that the x axis has a linear scale.

## X. DISCUSSION AND CONCLUSIONS

In this paper we have investigated the non-linear  $\sigma$ -model given by eqs. (1), (2). This model can be interpreted as a classical statistical model. In this case  $\beta$  plays the role of an inverse temperature. This new class of non-linear  $\sigma$ -models shows several interesting features. For all temperatures below the critical temperature,  $\beta > \beta_c$ , we find a universal critical behavior. Our model is therefore an example for self-tuned criticality, where critical behavior occurs without a tuning of parameters. This contrasts with the usual situation where critical behavior occurs only for a special choice of parameters, e.g.  $\beta = \beta_c$ . The presence of long range correlations for all  $\beta > \beta_c$  shows certain analogies with the Kosterlitz-Thouless phase transi-

tion [11]. Our model belongs, however, to a new universality class that differs in important aspects from the universality class characterizing the Kosterlitz-Thouless phase transition. It is not restricted to an abelian symmetry  $SO(2)$  and shows a very different long distance behavior. It remains to be seen if some particular condensed matter system realizes this universality class.

A characteristic feature of our model is the stripe phase, where the symmetries of discrete reflections and  $\pi/2$ -rotations on the lattice are preserved only if they are combined with gauge transformations acting on the flavor indices of the scalar field. The stripes single out preferred axes for quantities that are not invariant under gauge transformations. The presence of a stripe phase is not mainly a property of the lattice formulation. We show in appendix D that a stripe phase is also present in a continuum model with effective action similar to eq. (2). Such an effective action is a candidate for the description of the universality class, while we have not yet made any detailed comparison of its properties with our numerical results. Nevertheless, we have established that the effective action discussed in appendix D can describe the first order phase transition from the disordered to the stripe phase.

The most prominent property of our non-linear  $\sigma$ -model is lattice diffeomorphism invariance. This implies general coordinate invariance or diffeomorphism symmetry in the continuum limit. Our main interest concerns the possible relations to quantum gravity and we therefore focus on collective order parameters that play the role of a metric and a zweibein. This realizes geometrical features for our statistical model, close to the conceptual framework of ref. [13]. We have computed expectation values as well as correlation functions for the collective fields using numerical simulations.

Our main findings are mentioned in the introduction and we may concentrate here on two aspects: (1) non-vanishing expectation values of the metric for the vacuum state, describing flat space with either Minkowski or euclidean signature, (2) long range correlations for the metric fluctuations that decay with  $r^{-2}$  as a function of euclidean distance  $r$ .

As in usual gravity, the geometry of space or spacetime is deformed by the presence of matter. This can be seen by introducing an energy momentum tensor as a source term for the metric fluctuations. The response of geometry to a local energy momentum tensor is fixed in the linear approximation by the metric correlation function. One finds for the deviation from flat space

$$h_{\mu\nu}(x) = \int_y G_{\mu\nu,\rho\sigma}(x,y) t^{\rho\sigma}(y), \quad (87)$$

with  $t^{\rho\sigma}$  an appropriate energy momentum density and  $G_{\mu\nu,\rho\sigma}$  the metric correlator defined in eq. (68). In particular, for  $t^{00}(y) = K\delta(y)$  and vanishing other components, one obtains

$$h_{\mu\nu}(x) = KG_{\mu\nu,00}(x,0). \quad (88)$$

For this source the metric perturbation  $h_{00}$ , that may be compared to the Newtonian potential in four dimen-

sional gravity, decays  $\sim r^{-2}$ , with euclidean distance  $r^2 = x^\mu x^\nu \delta_{\mu\nu}$ , as visible in Fig. 9.

A static localized massive object in a Minkowski setting would correspond to a time independent energy momentum tensor

$$t^{00}(y^0, y^1) = M\delta(y^1). \quad (89)$$

This results in a static metric that decays inversely proportional to the spatial distance  $|x_1|$ ,

$$h_{00}(x_0, x_1) \sim \frac{M}{|x_1|}, \quad (90)$$

similar to four dimensional gravity.

Our model can be considered as a model for two-dimensional quantum gravity in the sense that the effective action for the metric is invariant under general coordinate transformations and that the metric correlations are long range. As a perhaps surprising effect the vacuum state corresponds to flat space even in the presence of quantum fluctuations. There are, however, also important differences as compared to Einstein gravity in four dimensions. They are mainly related to the non-trivial stripe order for  $\beta > \beta_c$ . The presence of preferred axes may lead to features that are not encountered if all order parameters preserve Lorentz symmetry as for standard four-dimensional gravity. The stripe order is reflected in the behavior of the correlation functions for the zweibein. In appendix E we discuss a simple ansatz for the zweibein effective action in the presence of stripes. It seems to differ substantially from the simple Lorentz-invariant setting of appendix B. While the effective action for scalars and zweibein is assumed to be Lorentz invariant, the Lorentz symmetry may be spontaneously broken the vacuum by the stripe configuration.

Several important issues remain to be solved before a more realistic model for quantum gravity can be constructed from a suitable scalar field theory on a lattice. The approach using a collective vielbein seems quite promising. A diffeomorphism invariant lattice action can then easily be formulated by employing the determinant of the collective vielbein. In this case one would like to implement the Lorentz transformations acting on the internal or flavor index of the vielbein as an exact symmetry. This is possible along the lines discussed in appendix D. One may employ a Lorentz invariant potential  $V(\rho_1 \rho_2)$  already for the microscopic lattice action. The non-linear constraint (1) can then be replaced by a Lorentz invariant constraint, for example by a bound on the Lorentz invariant product  $\rho_1 \rho_2 \leq C$ . The second issue concerns the preservation of a global Lorentz symmetry for a flat space ground state. One possible solution is a non-zero expectation value for the vielbein  $e_\mu^m \sim \delta_\mu^m$ , while the order parameter for the stripe configuration vanishes. Finally, an important step is the transition from two to four dimensions.

Several of the mentioned problems are absent or solved in lattice spinor gravity. However, reliable computations are difficult for spinor gravity. For this reason it seems worthwhile to pursue in parallel the scalar approach to lattice gravity which permits relatively cheap numerical simulations.

## Appendix A: Propagating metric in two-dimensional gravity

In this appendix we demonstrate that two-dimensional gravity can have propagating metric degrees of freedom. The issue which degrees of freedom propagate depends on the form of the quantum effective action. We present a simple example for such an action where metric degrees of freedom are indeed propagating.

In two dimensions the curvature tensor has only one independent component that is related to the curvature scalar  $R$  by

$$R_{0101} = \frac{1}{2} \det(g_{\mu\nu}) R. \quad (A1)$$

The integral  $\int_x \sqrt{g} R$  is a topological invariant and does not contribute to the field equations of the metric. If only this term and a two-dimensional cosmological constant are present in the effective action there will be no kinetic term for the metric, such that the metric is not a propagating field.

Nevertheless, a diffeomorphism invariant effective action can be constructed as

$$\Gamma = \int_x \sqrt{g} R f(-D^2) R, \quad (A2)$$

where  $D^2 = D^\mu D_\mu$  and  $D_\mu$  is the covariant derivative. This can be generalized by adding terms with higher powers of  $R$ . We will concentrate on non-local invariants[14] of the type

$$\Gamma = c_\kappa I_\kappa, \quad I_\kappa = \int_x \sqrt{g} R (-D^2)^{-\kappa} R, \quad (A3)$$

with  $\kappa > 0$ . Our setting remains more general, however.

In linear order of an expansion around flat space,  $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ ,  $h = h_{\mu\nu} \eta^{\mu\nu}$  one finds (for arbitrary dimension  $d$ )

$$R = \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h = -\frac{d-1}{d} \partial^2 \zeta, \quad (A4)$$

with

$$\zeta = h - \frac{d}{d-1} \frac{\partial_\mu \partial_\nu}{\partial^2} \tilde{h}^{\mu\nu}, \quad h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{d} h \eta_{\mu\nu}. \quad (A5)$$

It is easy to check that  $\zeta$  is invariant under the inhomogeneous part of the gauge transformations,  $\delta_{\text{inh}} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ . For  $d = 2$  the invariant (A3) reads in quadratic order in  $h_{\mu\nu}$

$$I_\kappa = \frac{1}{4} \int_x \zeta (-\partial^2)^{2-\kappa} \zeta. \quad (A6)$$

For an effective action  $\Gamma = c_\kappa I_\kappa$ ,  $\kappa > 1$ , and for euclidean signature  $\eta_{\mu\nu} = \delta_{\mu\nu}$ , this implies correlation functions decoupling for large  $(x - y)$  as

$$\begin{aligned} \langle \zeta(x) \zeta(y) \rangle &\sim |x - y|^{2-2\kappa}, \\ \langle \partial^2 \zeta(x) \partial^2 \zeta(y) \rangle &\sim |x - y|^{-(2+2\kappa)}. \end{aligned} \quad (A7)$$



For Minkowski signature the field equation for  $\zeta$ , which is obtained by taking a functional derivative of the effective action (A6), describes a relativistic wave equation for a propagating degree of freedom.

A non-local effective action of the type (A3) typically indicates the presence of a massless degree of freedom. For a suitable choice of degrees of freedom,  $\Gamma$  can often be written in an equivalent (quasi-)local form. For example, an effective action

$$\Gamma = \int_x \sqrt{g} \left\{ -\frac{1}{2} \chi Z (-D^2) (-D^2) \chi + f \chi R \right\} \quad (\text{A8})$$

describes a scalar field  $\chi$  with non-trivial wave function renormalization  $Z$  of the kinetic term and local coupling to the curvature scalar  $R$ . This implies for the scalar field  $\chi$  the field equation

$$\chi = f Z^{-1} (-D^2) (-D^2)^{-1} R. \quad (\text{A9})$$

Insertion of eq. (A9) into eq. (A8) yields the gravitational effective action

$$\Gamma = \frac{f^2}{2} \int_x \sqrt{g} R Z^{-1} (-D^2) (-D^2)^{-1} R. \quad (\text{A10})$$

For  $Z = -D^2$  we recover the action (A3) with  $\kappa = 2$ . For an expansion around flat space the scalar  $\chi$  is directly related to  $\zeta$  in eq. (A5)

$$\chi = \frac{f}{2} Z^{-1} (-\partial^2) \zeta. \quad (\text{A11})$$

The scalar  $\chi$  or  $\zeta$  is the only propagating field in this type of two-dimensional gravity. Correlation functions for metric components with an overlap with this scalar field, e.g. for  $h_{00}$  or  $h_{11}$ , should show a powerlike decay given by eq. (A7).

The effective action (A3) is invariant under general coordinate transformations. This is reflected in the linear expansion by the fact that only the particular combination  $\zeta$  of metric components contributes to the action. Without diffeomorphism symmetry nothing particular distinguishes  $\zeta$  from the other components of  $h_{\mu\nu}$ . The expansion of the effective action (A3) around a flat space exhibits the symmetry of global rotations or global Lorentz transformations, depending on the signature of the metric. In this important aspect it differs from the effective action for the collective metric in our non-linear  $\sigma$ -model (2).

## Appendix B: Possible effective action for propagating zweibein in two dimensions

In this appendix we briefly discuss an example for a possible form of an effective action for the zweibein. This serves as an illustration of some of the effects that may be expected for a more realistic effective action. We insist on diffeomorphism symmetry of the effective action, but we do not require local Lorentz symmetry acting on the index  $m$  of the zweibein  $e_\mu^m$ . Similarly, we also do not

impose euclidean rotation symmetry. As a consequence, the covariant derivative contains no spin connection, and  $D_\rho e_\mu^m$  can differ from zero. (A discussion of this type of generalized geometry can be found in ref. [15]).

Our example for an effective action involves the determinant of a “renormalized zweibein”

$$e_\mu^{Rm} = Z_{\mu n}^{m\nu} e_\nu^n. \quad (\text{B1})$$

Here the “wave function renormalization”  $Z_{\mu n}^{m\nu}$  is a function of covariant derivatives  $D_\mu$  such that  $e_\mu^{Rm}$  transforms again as a covariant vector. An example is ( $D^2 = D_\mu D^\mu$ )

$$Z_{\mu n}^{m\nu} = z_1 (-D^2) \delta_\mu^\nu \delta_n^m + z_2 (-D^2) D_\mu D^\nu \delta_n^m, \quad (\text{B2})$$

with  $z_1$  and  $z_2$  scalar functions. (Covariant derivatives involve the Levi-Cevita connection in the usual way, but no spin connection.) For the leading term in the effective action our ansatz reads

$$\begin{aligned} \Gamma &= -\frac{\beta}{4\Delta^2} \epsilon^{\mu\nu} \epsilon_{mn} \int d^2x e_\mu^{Rm} e_\nu^{Rn} \\ &= -\frac{\beta}{4\Delta^2} \int d^2x A_{mn}^{\mu\nu} e_\mu^m e_\nu^n, \end{aligned} \quad (\text{B3})$$

with

$$A_{mn}^{\mu\nu} = Z_{\rho m}^{\tau\mu} Z_{\sigma n}^{s\nu} \epsilon^{\rho\sigma} \epsilon_{rs}. \quad (\text{B4})$$

For the example (B2) one has

$$\begin{aligned} A_{mn}^{\mu\nu} &= \{ z_1^2 \epsilon^{\mu\nu} + z_1 z_2 (D_\rho D^\mu \epsilon^{\rho\nu} - D_\rho D^\nu \epsilon^{\rho\mu}) \\ &\quad + z_2^2 D_\rho D^\mu D_\sigma D^\nu \epsilon^{\rho\sigma} \} \epsilon_{mn}. \end{aligned} \quad (\text{B5})$$

The effective action (B3) is diffeomorphism invariant provided that covariant derivatives of tensors are again tensors, such that  $A_{mn}^{\mu\nu}$  transforms as a scalar function multiplied by  $\epsilon^{\mu\nu}$ . The effective action would be invariant under a global generalized Lorentz transformation if the covariant and contravariant derivatives are singlets with respect to this transformation. This is possible only for one of the groups  $SO(2)$  or  $SO(1,1)$ , but not for both simultaneously. For example, a definition

$$D^\mu = e_m^\mu e_n^\nu \hat{\eta}^{mn} D_\nu = g^{\mu\nu} D_\nu \quad (\text{B6})$$

requires the specification of  $\hat{\eta}_{mn}$ , with inverse  $\hat{\eta}^{mn}$  obeying  $\hat{\eta}^{mn} \hat{\eta}_{np} = \delta_p^m$ . The contravariant derivative also involves the inverse zweibein  $e_m^\mu$  which is defined by the relations

$$e_m^\mu e_\mu^n = \delta_m^n, \quad e_\mu^m e_m^\nu = \delta_\mu^\nu. \quad (\text{B7})$$

In order to gain some intuition for the implications of the effective action (B3) we first consider the approximation where  $z_1$  is a function of  $-\partial^2 = -\partial_\mu \partial_\nu \hat{\eta}^{\mu\nu}$ , while  $z_2 = 0$ . The quantum field equation reads

$$\frac{\delta \Gamma}{\delta e_\mu^m} = -\frac{\beta z_1^2}{2\Delta^2} \epsilon^{\mu\nu} \epsilon_{mn} e_\nu^n = 0. \quad (\text{B8})$$

Flat space with  $e_\nu^n \sim \delta_\nu^n$  solves this field equation provided that the Fourier transform  $z_1(q)$  vanishes for vanishing momenta  $q = (q_0, q_1)$ . The second functional derivative reads in momentum space

$$\frac{\delta^2 \Gamma}{\delta e_\mu^m(-q) \delta e_\nu^n(q')} = -\frac{\beta z_1^2}{2\Delta^2} \epsilon_{mn} \epsilon^{\mu\nu} \delta(q, q'). \quad (\text{B9})$$

For  $z_1$  a function of  $q^2 = q^\mu q_\mu = -q_0^2 + q_1^2$  the corresponding inverse propagator is a Lorentz covariant expression.

In the space of the four-component vector  $E = (e_0^0, e_0^1, e_1^0, e_1^1)$  the inverse propagator takes the form (we omit the  $\delta$ -function in Fourier space which reflects translation symmetry)

$$G^{-1} = -\frac{\beta z_1^2}{2\Delta^2} B, \quad (\text{B10})$$

with  $B$  an invertible  $4 \times 4$  matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B^2 = 1. \quad (\text{B11})$$

The propagator in momentum space is therefore given by

$$G = -\frac{2\Delta^2}{\beta} z_1^{-2}(q^2) B. \quad (\text{B12})$$

The non-vanishing correlation functions (84) are

$$F_{01}^{01} = F_{10}^{10} = -F_{01}^{10} = -F_{10}^{01} = -\frac{2\Delta^2}{\beta} z_1^{-2}(q^2). \quad (\text{B13})$$

This clearly differs from the observed structure of zweibein correlations for our non-linear  $\sigma$ -model, as discussed in sect. IX. It becomes clear however, that very different structures can also be accounted for by the ansatz (B3). In appendix E we discuss a possible form of an effective action for scalars and zweibein that may be somewhat closer to our model.

### Appendix C: Zweibein correlations from scalar correlations

For large  $\beta$  one expects that the zweibein is well approximated by the scalar expectation values  $\langle \psi \rangle$ ,

$$e_\mu^m = 2\Delta \langle \psi_\alpha \rangle \sigma_{\alpha\beta}^m \partial_\mu \langle \psi_\beta \rangle. \quad (\text{C1})$$

Similarly, one may try to approximate the zweibein fluctuations by the scalar fluctuations,  $\delta\psi_\alpha = \psi_\alpha - \langle \psi_\alpha \rangle$ , by linearizing eq. (58),

$$f_\mu^m = \tilde{e}_\mu^m - e_\mu^m = 2\Delta \{ \delta\psi_\alpha \sigma_{\alpha\beta}^m \partial_\mu \langle \psi_\beta \rangle + \langle \psi_\alpha \rangle \sigma_{\alpha\beta}^m \partial_\mu \delta\psi_\beta \}. \quad (\text{C2})$$

In this approximation the zweibein correlations are approximated by the scalar correlations

$$H_{\alpha\beta}(x, y) = \langle \delta\psi_\alpha(x) \delta\psi_\beta(y) \rangle, \quad (\text{C3})$$

namely

$$\begin{aligned} F_{\mu\nu}^{mn}(x, y) &= 4\Delta^2 \sigma_{\alpha\beta}^m \sigma_{\gamma\delta}^n \left\{ \partial_\mu \langle \psi_\beta(x) \rangle \partial_\nu \langle \psi_\delta(y) \rangle \right. \\ &\quad - \langle \psi_\beta(x) \rangle \partial_\nu \langle \psi_\delta(y) \rangle \frac{\partial}{\partial x^\mu} - \partial_\mu \langle \psi_\beta(x) \rangle \langle \psi_\delta(y) \rangle \frac{\partial}{\partial y^\nu} \\ &\quad \left. + \langle \psi_\beta(x) \rangle \langle \psi_\delta(y) \rangle \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right\} H_{\alpha\gamma}(x, y). \end{aligned} \quad (\text{C4})$$

The relation (C4) between  $F_{\mu\nu}^{mn}$  and  $H_{\alpha\gamma}$  involves the expectation values of the scalar fields. We will evaluate them for the stripe configuration (21), (23). Since we are interested in large separations  $(x - y)$  we employ the continuum limit. In the complex formulation we take for the cell averages continuous fields

$$\begin{aligned} \langle \varphi_1(x) \rangle &= \frac{1}{2\sqrt{2}} (1 - i) \exp \left\{ -\frac{i\pi}{4\Delta} (x^0 + x^1) \right\}, \\ \langle \varphi_2(x) \rangle &= \frac{1}{2\sqrt{2}} (1 - i) \exp \left\{ -\frac{i\pi}{4\Delta} (-x^0 + x^1) \right\}. \end{aligned} \quad (\text{C5})$$

These fields are indeed invariant under the combined translations  $t_0$  and  $t_1$ , cf. eqs. (25), (26). Taking partial derivatives of eq. (C5) reproduces the relations (28) up to a factor  $\pi/4$ . This conversion factor for derivatives between the discrete and continuum formulation (e.g. discrete derivatives involving finite distances versus continuous derivatives, and cell averages versus continuous fields) has to be applied to eqs. (C1), (C2), (C4) if we use the standard partial derivatives, i.e.  $\partial_\mu \rightarrow (4/\pi) \partial_\mu$ . (The factor  $4\Delta^2$  in eq. (C4) gets replaced by  $(64/\pi^2) \Delta^2$  and the factor  $2\Delta$  in eqs. (C1), (C2) becomes  $8\Delta/\pi$ . With this replacement the evaluation of eq. (C1) for the stripe configuration (C5) yields indeed  $e_\mu^m = \delta_\mu^m$ .) We also observe the normalization  $\langle \varphi_1^* \rangle \langle \varphi_1 \rangle = \langle \varphi_2^* \rangle \langle \varphi_2 \rangle = 1/4$ .

We finally take into account that the normalization of the expectation value differs from eq. (1) by multiplying  $\langle \varphi_a \rangle$  by a factor  $Z_\varphi$ , such that  $|\langle \varphi_1 \rangle|^2 + |\langle \varphi_2 \rangle|^2 = Z_\varphi^2 \leq 1$ . In terms of the real fields  $\langle \psi_\alpha \rangle$  the stripe configuration becomes

$$\begin{aligned} \langle \psi_1 \rangle &= \frac{Z_\varphi}{2\sqrt{2}} \left[ \cos \left( \frac{\pi}{4\Delta} (x^0 + x^1) \right) - \sin \left( \frac{\pi}{4\Delta} (x^0 + x^1) \right) \right], \\ \langle \psi_2 \rangle &= \frac{Z_\varphi}{2\sqrt{2}} \left[ \cos \left( \frac{\pi}{4\Delta} (-x^0 + x^1) \right) - \sin \left( \frac{\pi}{4\Delta} (-x^0 + x^1) \right) \right], \\ \langle \psi_3 \rangle &= -\frac{Z_\varphi}{2\sqrt{2}} \left[ \sin \left( \frac{\pi}{4\Delta} (x^0 + x^1) \right) + \cos \left( \frac{\pi}{4\Delta} (x^0 + x^1) \right) \right], \\ \langle \psi_4 \rangle &= -\frac{Z_\varphi}{2\sqrt{2}} \left[ \sin \left( \frac{\pi}{4\Delta} (-x^0 + x^1) \right) + \cos \left( \frac{\pi}{4\Delta} (-x^0 + x^1) \right) \right]. \end{aligned} \quad (\text{C6})$$

Inserting eq. (C6) into eq. (C4) yields explicit expressions for the zweibein correlations as linear combinations of the scalar correlations. So far, we have not attempted to check this type of relations numerically. The different qualitative behavior of scalar and zweibein correlations sheds doubts on the validity of such an approximation.

### Appendix D: Effective action for scalars and field equations

The quantum effective action  $\Gamma[\psi]$  for the scalar fields is defined in the usual way by introducing sources for the scalar fields in the functional integral, and performing a Legendre transform of the generating functional for the connected Greens functions. It includes all effects of fluctuations and generates the one-particle-irreducible Greens functions. Thus the functions for an arbitrary number of fields follow from  $\Gamma$  by simple functional differentiation. In this sense the knowledge of  $\Gamma$  amounts to a solution of the model. We do not attempt here a computation of the effective action. We rather investigate a simple ansatz which respects the symmetries of our model, namely

$$\Gamma = \frac{1}{2} \epsilon_{\mu\nu} \epsilon^{mn} \int d^2x V(\psi) \psi_\alpha \sigma_{\alpha\beta}^m \partial_\mu \psi_\beta \psi_\gamma \sigma_{\gamma\delta}^n \partial_\nu \psi_\delta, \quad (D1)$$

where the  $\sigma_{\alpha\beta}^m$  matrices are defined in eq. (59). (We use  $\psi_\alpha$  instead of  $\langle\psi_\alpha(x)\rangle$  in the following.) Eq. (D1) equals the classical action (2) for  $V(\psi) = -\beta$ . However, we admit here a general “scalar potential”  $V(\psi)$ . If  $V$  depends only on  $\rho_1 = \varphi_1^* \varphi_1 = \psi_1^2 + \psi_3^2$  and  $\rho_2 = \varphi_2^* \varphi_2 = \psi_2^2 + \psi_4^2$ , with  $V(\rho_1, \rho_2) = V(\rho_2, \rho_1)$ , the effective action shares all symmetries of the classical action. We use the continuum version of the effective action (D1) in order to demonstrate that the phase transition to the stripe phase also occurs in a continuum theory.

The vacuum state (or thermal equilibrium state in case of a classical statistical interpretation) is a solution of the quantum field equations. In our case, the quantum field equations for the scalar fields are obtained from the functional derivative of the effective action (D1). In the absence of sources they read

$$\begin{aligned} \frac{\delta\Gamma}{\delta\psi_\alpha} &= \epsilon_{\mu\nu} \epsilon^{mn} \{ \psi_\gamma \sigma_{\gamma\delta}^n \partial_\nu \psi_\delta \\ &\times [2V \sigma_{\alpha\beta}^m \partial_\mu \psi_\beta + V'_\alpha \psi_\eta \sigma_{\eta\beta}^m \partial_\mu \psi_\beta \psi_\alpha] \\ &+ \partial_\mu [V \psi_\gamma \sigma_{\gamma\delta}^n \partial_\nu \psi_\delta] \sigma_{\alpha\beta}^m \psi_\beta \} = 0. \end{aligned} \quad (D2)$$

Here we use

$$\frac{\partial V}{\partial \psi_\alpha} = 2V'_\alpha \psi_\alpha, \quad (D3)$$

with  $V'_\alpha = \partial V / \partial \rho_1$  for  $\alpha = 1, 3$ ,  $V'_\alpha = \partial V / \partial \rho_2$  for  $\alpha = 2, 4$ . The field equations always admit the solution  $\psi_\alpha = 0$  which corresponds to the disordered phase. We are interested here in the stripe solutions which are obtained for

$$2V \sigma_{\alpha\beta}^m \partial_\mu \psi_\beta \epsilon^{\mu\nu} \epsilon_{mn} \hat{e}_\nu{}^n = -2V'_\alpha \hat{e} \psi_\alpha, \quad (D4)$$

with  $V, V'_\alpha$  and

$$\hat{e}_\mu{}^m = \psi_\gamma \sigma_{\gamma\delta}^m \partial_\mu \psi_\delta, \quad \hat{e} = \det(\hat{e}_\mu{}^m), \quad (D5)$$

independent of  $x$ . Multiplication of eq. (D4) by  $\psi_\alpha$  and summing over  $\alpha$  yields as a condition for the existence of this type of solution

$$\rho_1 \frac{\partial V}{\partial \rho_1} + \rho_2 \frac{\partial V}{\partial \rho_2} = -2V. \quad (D6)$$

We make the ansatz

$$\partial_\mu \psi_\beta = (U_\mu)_{\beta\gamma} \psi_\gamma, \quad (D7)$$

where only the elements  $(1, 3), (3, 1), (2, 4)$  and  $(4, 2)$  of the constant matrices  $U_0$  and  $U_1$  differ from zero. Eq. (D4) is obeyed for  $\psi_\alpha \neq 0$  if

$$\epsilon_{\mu\nu} \epsilon^{mn} \hat{e}_\nu{}^n (2V u_\mu^{\alpha,m} + V'_\alpha \hat{e}_\mu{}^m) = 0. \quad (D8)$$

Here  $u_\mu^{\alpha,m}$  is defined by the condition

$$\sigma_{\alpha\beta}^m (U_\mu)_{\beta\gamma} = u_\mu^{\alpha,m} \delta_{\alpha\gamma} \quad (D9)$$

and obeys

$$\hat{e}_\mu{}^m = \sum_{\alpha=1}^4 u_\mu^{\alpha,m} \psi_\alpha^2. \quad (D10)$$

The squared matrices  $(U_\mu)^2$  are diagonal

$$(U_\mu^2)_{\alpha\beta} = \zeta_{\mu,\alpha} \delta_{\alpha\beta}, \quad \zeta_{\mu,1} = \zeta_{\mu,3}, \quad \zeta_{\mu,2} = \zeta_{\mu,4}. \quad (D11)$$

Our ansatz (D7) requires then

$$(\partial_\mu)^2 \psi_\alpha = \zeta_{\mu,\alpha} \psi_\alpha. \quad (D12)$$

We will require the matrices  $U_\mu$  to be antisymmetric, guaranteeing  $\partial_\mu \rho_1 = \partial_\mu \rho_2 = 0$ . Then the coefficients  $\zeta_{\mu,\alpha}$  are negative (or zero). As a consequence, one obtains solutions of eq. (D12) which are periodic in both  $x^0$  and  $x^1$ . We typically will find solutions with  $\zeta_{0,\alpha} = \zeta_{1,\alpha}$ . They obey the wave equation

$$(\partial_0^2 - \partial_1^2) \psi_\alpha = 0. \quad (D13)$$

It is remarkable how wave equations with two-dimensional Lorentz symmetry arise in a natural way from the field equations derived from the action (D1).

In analogy with eq. (C6) we consider possible solutions of the type

$$\begin{aligned} \psi_1 &= c_1 [\cos(P_1(x^0 + x^1)) - \sin(P_1(x^0 + x^1))] \\ \psi_3 &= -c_1 [\sin(P_1(x^0 + x^1)) + \cos(P_1(x^0 + x^1))] \\ \psi_2 &= c_2 [\cos(P_2(-x^0 + x^1)) - \sin(P_2(-x^0 + x^1))] \\ \psi_4 &= -c_2 [\sin(P_2(-x^0 + x^1)) + \cos(P_2(-x^0 + x^1))], \end{aligned} \quad (D14)$$

with  $\rho_1 = 2c_1^2, \rho_2 = 2c_2^2$  and antisymmetric matrices  $U_\mu$  obeying

$$\begin{aligned} (U_0)_{13} &= P_1, \quad (U_1)_{13} = P_1, \\ (U_0)_{24} &= -P_2, \quad (U_1)_{24} = P_2, \\ \zeta_{0,1} &= \zeta_{1,1} = -P_1^2, \quad \zeta_{0,2} = \zeta_{1,2} = -P_2^2. \end{aligned} \quad (D15)$$

For  $\alpha = 1, 3$  one finds for all  $\mu$  and  $m$  that  $u_\mu^{\alpha,m} = P_1$  whereas for  $\alpha = 2, 4$  one has  $u_\mu^{\alpha,m} = P_2$  if  $\mu = m$ , and  $u_\mu^{\alpha,m} = -P_2$  if  $\mu \neq m$ . Eq. (D10) yields

$$\hat{e}_\mu{}^m = \begin{pmatrix} P_1 \rho_1 + P_2 \rho_2 & P_1 \rho_1 - P_2 \rho_2 \\ P_1 \rho_1 - P_2 \rho_2 & P_1 \rho_1 + P_2 \rho_2 \end{pmatrix}. \quad (D16)$$

For the particular stripe solution (C6) with  $P_1 = P_2 = \pi/(4\Delta)$ ,  $\rho_1 = \rho_2 = Z_\varphi^2/4$  one recovers

$$\hat{e}_\mu^m = \frac{Z_\varphi^2}{8\Delta} \delta_\mu^m, \quad e_\mu^m = \frac{8\Delta}{\pi} \hat{e}_\mu^m = Z_\varphi^2 \delta_\mu^m. \quad (\text{D17})$$

For general  $P_1, P_2, \rho_1, \rho_2$  one has

$$\hat{e} = \det(\hat{e}_\mu^m) = 4P_2 P_2 \rho_1 \rho_2, \quad (\text{D18})$$

and eq. (D8) is obeyed for

$$\rho_1 \frac{\partial V}{\partial \rho_1} = \rho_2 \frac{\partial V}{\partial \rho_2} = -V. \quad (\text{D19})$$

If eq. (D19) has a solution for suitable values of  $\rho_1$  and  $\rho_2$  we therefore find solutions with arbitrary  $P_1$  and  $P_2$ . The condition (D19) implies the condition (D6). We conclude that for a potential which admits a solution of eq. (D19) stripe solutions (D14) exists with arbitrary “momenta”  $P_1$  and  $P_2$ .

We may try to interpret the effective action (D1) as an approximation to the continuum limit of the quantum effective action which corresponds to the microscopic lattice action (2). In this case one expects a dependence of the shape of  $V$  on the parameter  $\beta$ . (This extends to a parameter dependence of  $V$  for other models in the same universality class.) A phase transition from the disordered phase with  $\psi_\alpha = 0$  to the stripe phase occurs at  $\beta_c$  if for  $\beta > \beta_c$  eq. (D19) has a solution and if for the corresponding stripe solution (D14) the action (D1) becomes negative. At the phase transition for  $\beta = \beta_c$  the effective action in the stripe phase vanishes, such that the free energy  $\Gamma$  has the same value for the disordered and the stripe phase. A first order transition is realized if for  $\beta = \beta_c$  the stripe solution still has a nonvanishing “order parameter”  $\psi_\alpha$ .

In the remainder of this appendix we discuss simple shapes of the potential  $V$  that realize the first order phase transition that we observe in our numerical results. A constant potential, e.g.  $V = -\beta$ , does not admit solutions with  $\psi_\alpha \neq 0$ . This has a simple explanation: the action is then a pure quartic polynomial of  $\psi$ , such that for any value of  $\psi$  for which  $\Gamma < 0$  the rescaled field  $(1+\epsilon)\psi$ ,  $\epsilon > 0$ , leads to an even smaller value of  $\Gamma$ , thus excluding an extremum for  $\Gamma \neq 0$ . For the microscopic action this problem is cured by the non-linear constraint (1), which would be translated to the continuum language as  $\rho_1 + \rho_2 = 1/2$ .

Solutions of the condition (D19) exist for a wide class of non-trivial potentials  $V(\rho_1, \rho_2)$  without invoking constraints for  $\rho_1$  and  $\rho_2$ . As a first example we consider a potential  $V(\rho)$ ,  $\rho = \rho_1 + \rho_2$ . The stripe solutions correspond then to an extremum of the combination

$$W(\rho) = \rho^2 V(\rho), \quad \frac{\partial W}{\partial \rho}(\rho_0) = 0. \quad (\text{D20})$$

Indeed, eq. (D20) implies that the condition (D19) has a solution with  $\rho_1 = \rho_2 = \rho_0/2$ . (For  $\beta \rightarrow \infty$  one would expect  $\rho_0 \rightarrow 1/2$ .) The value of the effective action for stripe solutions with  $\rho_1 = \rho_2 = \rho_0/2$  is given by  $W_0 =$

$W(\rho_0)$ ,

$$\Gamma_0 = \int d^2x \hat{e} V(\rho_0) = P_1 P_2 \int d^2x W_0. \quad (\text{D21})$$

For  $P_1 = P_2 = \pi/(4\Delta)$  the action per lattice point equals  $(\pi^2/8)W_0$ , such that for  $\beta \rightarrow \infty$  one expects  $W_0 \rightarrow -(8/\pi^2)\beta$ . We observe that  $\Gamma_0$  in eq. (D21) can be made arbitrarily negative for  $P_1 P_2 \rightarrow \infty$ . This “ultraviolet divergence” is cut off by the lattice regularization. We may consider  $(\pi/4\Delta)$  as the maximal momentum, say in the  $x^1$ -direction.

Many different forms of the potential  $V(\rho_1, \rho_2)$  are conceivable. For example, if  $V$  only depends on the combination  $\rho_1 \rho_2$  the effective action (D1) is invariant under local Lorentz transformations

$$\psi'_{1,3}(x) = e^{\alpha(x)/2} \psi_{1,3}(x), \quad \psi'_{2,4}(x) = e^{-\alpha(x)/2} \psi_{2,4}(x) \quad (\text{D22})$$

for which the zweibein transforms as

$$\begin{aligned} (\hat{e}_\mu^0)' &= \cosh \alpha \hat{e}_\mu^0 + \sinh \alpha \hat{e}_\mu^1, \\ (\hat{e}_\mu^1)' &= \cosh \alpha \hat{e}_\mu^1 + \sinh \alpha \hat{e}_\mu^0. \end{aligned} \quad (\text{D23})$$

For the example (with positive constants  $a, b$ )

$$V = -a + b\rho_1\rho_2 \quad (\text{D24})$$

one has  $\rho_1 \partial V / \partial \rho_1 = \rho_2 \partial V / \partial \rho_2 = b\rho_1\rho_2$  and eq. (D19) is met for

$$\rho_{1,0}\rho_{2,0} = \frac{a}{2b}, \quad (\text{D25})$$

with

$$V(\rho_{10}, \rho_{20}) = -\frac{a}{2}. \quad (\text{D26})$$

Indeed, the combination  $\hat{e}V$  takes for the configurations (D14) the form

$$\hat{e}V = 4P_1 P_2 [-a\rho_1\rho_2 + b(\rho_1\rho_2)^2]. \quad (\text{D27})$$

For  $P_1 P_2 > 0$  this has a minimum for eq. (D25) with  $\hat{e} > 0$ ,  $(\hat{e}V)_0 = -P_1 P_2 a^3/(2b^2)$ . We observe a degeneracy of the minimum under global Lorentz transformations (D22).

For  $P_1 P_2 < 0$  the combination  $\hat{e}V$  has a maximum with  $\hat{e} < 0$ ,  $\hat{e}V_0 > 0$ . The action can become arbitrarily negative for  $P_1 P_2 < 0$  and large  $\rho_1 \rho_2$ . We may prevent this to happen by imposing a Lorentz invariant constraint

$$\rho_1 \rho_2 < \frac{a}{b}. \quad (\text{D28})$$

An interesting issue concerns the possibility to use eq. (D1) with eq. (D24) for the microscopic action  $S$ , and to replace the non-linear constraint (1) by the condition (D28). This would permit to realize Lorentz symmetry as an exact symmetry of the model.

For our model the observed first order transition at  $\beta_c$  indicates an effective potential that is more complicated than the form (D24). Indeed, a first order transition is described if a possible local minimum of  $\hat{e}V$  with  $\rho_1, \rho_2$

different from zero occurs for  $\beta_c$  at  $(\hat{e}V)_0 = 0$ , while for  $\beta < \beta_c$  one has  $(\hat{e}V)_0 > 0$ . For  $\beta < \beta_c$  the absolute minimum of the effective action (D1) will then be found at  $\psi = 0$ , corresponding to the disordered phase.

As an example, consider

$$V = -a + b\rho_1\rho_2 - \frac{c}{2}(\rho_1 + \rho_2) + \frac{d}{4}(\rho_1 + \rho_2)^2. \quad (\text{D29})$$

Possible solutions of eq. (D19) with  $\rho_1 = \rho_2 = \rho/2$  occur for

$$\rho_0 = \frac{3c \pm \sqrt{9c^2 + 32a(d+b)}}{4(d+b)} \quad (\text{D30})$$

with

$$V_0 = V(\rho_0) = -\frac{a}{2} - \frac{1}{8}c\rho_0. \quad (\text{D31})$$

For  $d+b > 0$  and  $c > 0$  one finds a critical value

$$a_c = -\frac{c^2}{4(d+b)}, \quad (\text{D32})$$

such that for  $a > a_c$  one has  $V_0 < 0$ , and for  $a < a_c$  the solution (D30) with smallest  $V$  occurs for  $V_0 > 0$ . Thus for  $a < a_c$  the disordered phase with  $\psi = 0$  is realized and we may associate the critical value  $\beta_c$  with  $a(\beta_c) = a_c$ . Indeed, for  $\rho_1 = \rho_2 = \rho/2$  we can consider

$$W(\rho) = \rho^2 V(\rho) = -a\rho^2 - \frac{c}{2}\rho^3 + \frac{1}{4}(d+b)\rho^4. \quad (\text{D33})$$

For  $a < 0$  one has a local minimum of  $W(\rho)$  at  $\rho = 0$ . A second local minimum exists for  $c > (4/3)\sqrt{-2a(d+b)}$ . For  $c > 2\sqrt{-a(d+b)}$ , corresponding to  $a > a_c$ , this second minimum occurs for negative  $W$  and is therefore deeper than the minimum at  $\rho = 0$ . At the critical value  $a_c$  the order parameter  $\sim \rho_0$  jumps from zero to

$$\rho_{0,c} = \frac{2c}{d+b}. \quad (\text{D34})$$

Having found a satisfactory description of the first order phase transition from the disordered phase to the stripe phase one may ask if a suitable shape of  $V$  can also account for the scalar correlation functions in the continuum limit. In principle, the inverse scalar correlation functions can be obtained from the second variation of the effective action, evaluated for the appropriate solution of the field equation (D2). We have not yet performed a computation of the correlation functions that correspond to the effective action (D1).

## Appendix E: Effective action for zweibein and scalars

The effective action for scalars and zweibein can be defined by introducing appropriate sources

$$\begin{aligned} W[\eta, t] &= \ln \int \mathcal{D}\tilde{\psi} \exp\{-S + \sum_{\tilde{y}} t_m^\mu(\tilde{y})\tilde{e}_\mu^m(\tilde{y}) \\ &\quad + \sum_{\tilde{z}} \eta_\alpha(\tilde{z})\tilde{\psi}_\alpha(\tilde{z})\}, \\ \frac{\partial W}{\partial t_m^\mu(\tilde{y})} &= \langle \tilde{e}_\mu^m(\tilde{y}) \rangle = e_\mu^m(\tilde{y}), \\ \frac{\partial W}{\partial \eta_\alpha(\tilde{z})} &= \langle \tilde{\psi}_\alpha(\tilde{z}) \rangle = \psi_\alpha(\tilde{z}), \end{aligned} \quad (\text{E1})$$

and performing a Legendre transform

$$\Gamma[\psi, e] = -W + \sum_{\tilde{y}} t_m^\mu(\tilde{y})e_\mu^m(\tilde{y}) + \sum_{\tilde{z}} \eta_\alpha(\tilde{z})\psi_\alpha(\tilde{z}). \quad (\text{E2})$$

This yields the exact quantum field equations

$$\frac{\partial \Gamma}{\partial e_\mu^m(\tilde{y})} = t_m^\mu(\tilde{y}), \quad \frac{\partial \Gamma}{\partial \psi_\alpha(\tilde{z})} = \eta_\alpha(\tilde{z}). \quad (\text{E3})$$

The scalar effective action  $\Gamma[\psi]$  discussed in the preceding section is obtained for  $t_m^\mu(\tilde{y}) = 0$ . It can be inferred from  $\Gamma[\psi, e]$  by solving the field equation  $\partial \Gamma / \partial e_\mu^m(\tilde{y}) = 0$  with solution  $e_\mu^{(0)m}(\tilde{y})[\psi]$  being a functional of  $\psi$ . Then  $\Gamma[\psi] = \Gamma[\psi, e^{(0)}[\psi]]$ .

Due to lattice diffeomorphism invariance of the action the continuum limit of the effective action is invariant under general coordinate transformations[8]. Besides diffeomorphism symmetry the effective action also preserves the discrete reflection symmetries of the lattice action as well as the continuous flavor symmetry.

Let us try an ansatz for the continuum limit of the effective action which is consistent with the symmetries

$$\begin{aligned} \Gamma &= \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{mn} \int d^2x \{ \psi_\alpha \sigma_{\alpha\beta}^m \partial_\mu \psi_\beta [V_1(\psi) \psi_\gamma \sigma_{\gamma\delta}^n \partial_\nu \psi_\delta \\ &\quad + V_2(\psi) e_\nu^n] + V_3(\psi) e_\mu^m e_\nu^m \}. \end{aligned} \quad (\text{E4})$$

The field equation for the zweibein reads

$$\frac{1}{2}\epsilon_{\mu\nu}\epsilon^{mn}(2V_3 e_\nu^m + V_2 \psi_\gamma \sigma_{\gamma\delta}^n \partial_\nu \psi_\delta) = t_m^\mu. \quad (\text{E5})$$

For  $V_{2,3} \neq 0$ , and in the absence of sources  $t_m^\mu = 0$ , the solution is

$$e_\nu^n = -\frac{V_2}{2V_3} \psi_\gamma \sigma_{\gamma\delta}^n \partial_\nu \psi_\delta. \quad (\text{E6})$$

The proportionality between  $e_\nu^n$  and  $\psi_\gamma \sigma_{\gamma\delta}^n \partial_\nu \psi_\delta$  is realized in our model for  $\beta > \beta_c$ , as can be seen in Fig. 8. The proportionality factor  $N^{(e)}/N^{(e\varphi)}$  corresponds to  $-V_2/2V_3$ , evaluated for the appropriate  $\rho_1\rho_2$  (cf. appendix D), and taking the proper normalization of partial derivatives into account.

Insertion of eq. (E6) into eq. (E4) yields eq. (D1) with

$$V = V_1 - \frac{V_2^2}{4V_3}. \quad (\text{E7})$$

The field equation for the scalars (E3) for  $\eta_\alpha = 0$  yield, after insertion of the solution (E6), precisely eq. (D2) with  $V$  given by eq. (E7). We can therefore take over the discussion of the preceding section. In particular, for  $\beta > \beta_c$

one finds the wave solutions (D14), with

$$\psi_\alpha \sigma_{\alpha\beta}^m \partial_\mu \psi_\beta \sim \delta_\mu^m, \quad e_\mu^m \sim \delta_\mu^m. \quad (\text{E8})$$

Eq. (E6) explains our finding  $e_\mu^m \sim \delta_\mu^m$  for all  $\beta > \beta_c$ , and that  $e_\mu^m$  as well as  $\psi$  vanish simultaneously for  $\beta < \beta_c$ . We realize a flat space geometry without tuning of parameters for all  $\beta > \beta_c$ .

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